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Overview of The Literacy and Numeracy Secretariat Professional Learning Series

The effectiveness of traditional professional development seminars and workshops has increasingly been questioned by both educators and researchers (Fullan, 1995; Guskey & Huberman, 1995; Wilson & Berne, 1999). Part of the pressure to rethink traditional PD comes from changes in the teaching profession. The expert panel reports for primary and junior literacy and numeracy (Ministry of Education, 2003, 2004) raise several key issues for today’s teachers:

- Teachers are being asked to teach in ways that they themselves may not have experienced or seen in classroom situations.
- Teachers require a more extensive knowledge of literacy and numeracy than they did previously as teachers or as students.
- Teachers need to develop a deep knowledge of literacy and numeracy pedagogy in order to understand and develop a repertoire of ways to work effectively with a range of students.
- Teachers may experience difficulty allocating sufficient time for students to develop concepts of literacy and numeracy if they themselves do not appreciate the primacy of conceptual understanding.

For professional learning in literacy and numeracy to be meaningful and classroom-applicable, these issues must be addressed. Effective professional learning for today’s teachers should include the following features:

- It must be grounded in inquiry and reflection, be participant-driven, and focus on improving planning and instruction.
- It must be collaborative, involving the sharing of knowledge and focusing on communities of practice rather than on individual teachers.
- It must be ongoing, intensive, and supported by a job-embedded professional learning structure, being focused on the collective solving of specific problems in teaching, so that teachers can implement their new learning and sustain changes in their practice.
- It must be connected to and derived from teachers’ work with students – teaching, assessing, observing, and reflecting on the processes of learning and knowledge production.

Traditionally, teaching has been a very isolated profession. Yet research indicates that the best learning occurs in collaboration with others (Fullan, 1995; Joyce & Showers, 1995; Staub, West & Miller, 1998). Research also shows that teachers’ skills, knowledge, beliefs, and understandings are key factors in improving the achievement of all students.

Job-embedded professional learning addresses teacher isolation by providing opportunities for shared teacher inquiry, study, and classroom-based research. Such collaborative professional
learning motivates teachers to act on issues related to curriculum programming, instruction, assessment, and student learning. It promotes reflective practice and results in teachers working smarter, not harder. Overall, job-embedded professional learning builds capacity for instructional improvement and leadership.

There are numerous approaches to job-embedded professional learning. Some key approaches include: co-teaching, coaching, mentoring, teacher inquiry, and study.

**Aims of Numeracy Professional Learning**

The Literacy and Numeracy Secretariat developed this professional learning series in order to:

- promote the belief that all students have learned some mathematics through their lived experiences in the world and that the mathematics classroom should be a place where students bring that thinking to work;
- build teachers’ expertise in setting classroom conditions in which students can move from their informal mathematics understandings to generalizations and formal mathematical representations;
- assist educators working with teachers of students in the junior division to implement the student-focused instructional methods that are referenced in *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6: Volume 2* to improve student achievement; and
- have teachers experience mathematical problem solving – sharing their thinking and listening; considering the ideas of others; adapting their thoughts; understanding and analysing solutions; comparing and contrasting solutions; and discussing, generalizing, and communicating – as a model of what effective mathematics instruction entails.

**Teaching Mathematics through Problem Solving**

Until quite recently, understanding the thinking and learning that the mind makes possible has remained an elusive quest, in part because of a lack of powerful research tools. In fact, many of us learned mathematics when little was known about learning or about how the brain works. We now know that mathematics instruction can be developmentally appropriate and accessible for today’s learners. Mathematics instruction has to start from contexts that children can relate to – so that they can “see themselves” in the context of the question. Most people learned mathematics procedures first and then solved word problems related to the operations after practising the skills taught to them by the teacher. The idea of teaching through problem solving turns this process on its head.

By starting with a problem in a context (e.g., situational, inquiry-based) that children can relate to, we activate their prior knowledge and lived experiences and facilitate their access to solving mathematical problems. This activation connects children to the problem; when they can make sense of the details, they can engage in problem solving. Classroom instruction needs to provoke students to further develop their informal mathematical knowledge by representing their mathematical thinking in different ways and by adapting their
understandings after listening to others. As they examine the work of other students and consider the teacher’s comments and questions, they begin to: recognize patterns; identify similarities and differences between and among the solutions; and appreciate more formal methods of representing their thinking. Through rich mathematical discourse and argument, students (and the teacher) come to see the mathematical concepts expressed from many points of view. The consolidation that follows from such dynamic discourse makes the mathematical representations explicit and lets students see many aspects and properties of mathematics concepts, resulting in students’ deeper understanding.

**Learning Goals of the Module**

This module is organized to guide facilitators as they engage participants in discussion with colleagues working in junior classrooms. This discourse will focus on understanding what teaching and learning mathematics through problem solving looks, sounds, and feels like. The use of particular high-yield strategies such as think-pair-share and bansho will be practised. As well, the modules may be used by facilitators as learning plans, to lead professional learning sessions in 4 parts – A, B, C, and D – of 60 min to 75 min each. They can also be used over several weeks or months as a job-embedded study. Each of the 4 sections A to D can stand alone and be the focus of a study session wherever and whenever it is appropriate to have staff meet and work together to study instructional strategies to improve mathematics achievement.

During these sessions, participants will:

- become familiar with the notion of learning *mathematics for teaching* as a focus for numeracy professional learning;
- experience learning mathematics through problem solving;
- solve problems in different ways; and
- develop strategies for teaching mathematics through problem solving.
Getting Organized

Participants

• Classroom teachers (experienced, new to the grade, new to teaching [NTIP]), resource and special education teachers, numeracy coaches, system curriculum staff, and school leaders will bring a range of experiences – and comfort levels – to the teaching and learning of mathematics. Participants may be organized by grade, division, cross-division, family of school clusters, superintendency regions, coterminous boards, or boards in regions.

• Adult learners benefit from a teaching and learning approach that recognizes their mathematics teaching experiences and knowledge and that provides them with learning experiences that challenge their thinking and introduces them to research-supported methods for teaching and learning mathematics. For example, if time permits, begin each session with 10 minutes for participants to share their mathematics teaching and learning experiences, strategies, dilemmas, and questions.

• Some participants may have prior knowledge through having attended professional development sessions using *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 3* or *The Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6* through board sessions or Ontario Summer Institutes. These professional learning sessions are intended to deepen numeracy learning, especially for junior teachers.

Facilitators

Effective professional learning happens daily and over time. These professional learning materials are designed to be used to facilitate teachers’ collaborative study of a particular aspect of *mathematics for teaching* to improve their instruction. These materials were not designed as presentation material. In fact, these sessions are organized so that they can be used flexibly with teachers (e.g., classroom teachers, coaches, consultants) and school leaders (e.g., vice principals, principals, program coordinators) to plan and facilitate their own professional learning at the school, region, and/or board levels.

It is recommended that the use of these materials be facilitated collaboratively by at least two educators. Co-facilitators have the opportunity to co-plan, co-implement, and make sense of the audience’s responses together, to adjust their use of the materials, and to improve the quality of the professional learning for the audience and themselves. Further, to use these modules, facilitators do not need to be numeracy experts, but facilitators do need to be confident about learning collaboratively with the participants and have some experience and/or professional interest in studying mathematics teaching/learning to improve instruction.

Here are a few ways that facilitators can prepare to use this module effectively:

• Take sufficient time to become familiar with the content and the intended learning process inherent in these sessions.

• Think about the use of the PowerPoint as a visual aid to present the mathematical prompts and questions participants will use.
• Use the Facilitator’s Handbook to determine ways in which to use the slides to generate discussion, mathematical thinking and doing, and reflection about classroom implementation.

• Note specific teaching strategies that are suggested to develop rich mathematical conversation or discourse.

• Highlight the mathematical vocabulary and symbols that need to be made explicit during discussions and sharing of mathematical solutions in the Facilitator’s Handbook.

• Try the problems prior to the sessions to anticipate a variety of possible mathematical solutions.

• As you facilitate the sessions, use the Facilitator’s Handbook to help you and your learning group make sense of the mathematical ideas, representations (e.g., arrays, number lines), and symbols.

**Time Lines**

• This module can be used in different professional learning scenarios: professional learning team meetings, teacher planning time, teacher inquiry/study, parent/community sessions.

• Though the module is designed to be done in its entirety, so that the continuum of mathematics learning can be experienced and made explicit, the sessions can be chosen to meet the specific learning needs of the audience. For example, participants may want to focus on understanding how students develop conceptual understanding of area through problem solving, so the facilitator may choose to implement only Session B in this module.

• As well, the time frame for implementation is flexible. Three examples are provided below.

<table>
<thead>
<tr>
<th>Module Sessions</th>
<th>One Full Day</th>
<th>Two Half-Days</th>
<th>Four Sessions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Session A – Activating Prior Knowledge</td>
<td>75 min</td>
<td>Day 1 120 – 180 min</td>
<td>90 – 120 min</td>
</tr>
<tr>
<td>Session B – Modelling and Representing Area</td>
<td>75 min</td>
<td>Day 2 120 – 180 min</td>
<td>90 – 120 min</td>
</tr>
<tr>
<td>Session C – Organizing and Coordinating Student Solutions to Problems Using Criteria</td>
<td>75 min</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Session D – Selecting and Writing Effective Mathematics Problems for Learning</td>
<td>75 min</td>
<td></td>
<td>90 – 120 min</td>
</tr>
</tbody>
</table>
If you choose to use these materials during:

- One full day – the time line for each session is tight for implementation; monitor the use of time for mathematical problem solving, discussion, and reflecting.
- Two half days – the time line for each session is tight for implementation; monitor the use of time for mathematical problem solving, discussion, and reflecting; include time for participants to share the impact of implementing ideas and strategies from the first half-day session.
- Four sessions – the time lines for each session are more generous for implementation; include time for participants to discuss and choose ideas and strategies to implement in their classroom at the end of each session; include time for sharing the impact of implementing ideas and strategies from the previous session at the start of each session.

**Creating a Professional Learning Environment**

- Organize participants into small groups – preferably of 4 to 6 people – to facilitate professional dialogue and problem-solving/thinking experiences.
- Seat participants in same-grade or cross-grade groups, depending on whether you want the discussion to focus on one grade level or across grade levels.
- Ensure that a blackboard or 3 m to 4 m of wall space is cleared, so that mathematical work can be posted and clearly seen.
- Provide a container with the learning materials (e.g., writing implements like markers, paper, sticky notes) on each table before the session. Math manipulatives and materials should be provided for each pair of participants at each table.
- Provide a copy of the agenda and handouts of the PowerPoint for note-taking purposes or tell the participants that the PowerPoint will be e-mailed to them after the session so that they have a record of it.
- Arrange refreshments for breaks and/or lunches, if appropriate.
- If time permits, begin each session with 10 min for participants to share their mathematics teaching and learning experiences, strategies, and dilemmas.

**Materials Needed**

- A Guide to Effective Instruction in Mathematics: Kindergarten to Grade 6, Volume 2: Problem Solving and Communication (Toronto: Queen’s Printer for Ontario, 2006)
- Teaching and Learning Through Problem Solving PowerPoint presentation, slides 1 to 65
- Ontario Curriculum, Grades 1 to 8: Mathematics (Toronto: Queen’s Printer for Ontario, 2005)
- computer, LCD projector, projector screen, extension cord
- cleared black board or white board
- highlighters and markers (6 markers of different colours for each table group)
• masking tape
• square grid chart paper (ripped into halves or quarters – about 5 quarter sheets per 3 participants)
• sticky notes
• transparencies and overhead markers (if projector is available),
• geoboards and geobands/elastics (1 set per 2 participants)
• metric ruler
• square tiles (at least 100 per table group), base ten blocks, calculators for every two participants
• LNS Deborah Loewenberg Ball webcast (see www.curriculum.org)
• BLM 1, Square Tile Paper (one per pair)
• BLM 2A, Size-It-Up Cards for Adults (one set for every 4 participants) and one envelope for each set
• BLM 2B, Size-It-Up Cards for Students
• BLM 3, Square Dot Paper (15 copies per pair)
• BLM 4, Assessment for Learning Seating Plan Tool (one per person)
• BLM 5, T-Charts for Comparing Ideas (one per person)
• BLM 6, Grades 3 to 6 Curriculum Connections for Area (one per person)
Session A – Activating Prior Knowledge

As stated in *Teaching and Learning Mathematics – The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario*, “instruction based on a problem-solving or investigative approach is the means by which Ontario students will most readily achieve mathematical literacy” (2004, p. 9). The recommendation that problem solving ought to be the mainstay of the mathematics program is also central to *A Guide to Effective Instruction in Mathematics: Kindergarten to Grade 6, Volume 2* (Ministry of Education, 2006).

According to pages 4–5 of the *Guide*:

In their everyday experiences, students are intuitively and naturally solving problems. They seek solutions to sharing toys with friends or building elaborate structures with construction materials. Teachers who use problem solving as the focus of their mathematics class help their students to develop and extend these intuitive strategies. Through relevant and meaningful experiences, students develop a repertoire of strategies and processes (e.g., steps for solving problems that they can apply when solving problems). Students develop their repertoire over time as they become more mature in their problem-solving skills. The problem-solving processes that Kindergarten students use will look very different from those that Grade 6 students use. Initially, students will rely on intuition. With exposure, experience, and shared learning, they will formalize an effective approach to solving problems by developing a repertoire of problem-solving strategies that they can use flexibly when faced with new problem-solving situations.

Problem solving needs to permeate the mathematical program rather than be relegated to a once-a-week phenomenon – the “problem of the week”. In this guide it is not considered to be one approach among many; rather, it is seen as the main strategy for teaching mathematics. Problem solving should be the mainstay of mathematical teaching and should be used daily.

Aims of Numeracy Professional Learning

Show slides 1 to 3. Assign each group of participants two aims to discuss. Ask the participants to explain how these aims of numeracy professional learning support teachers in improving their mathematics knowledge for teaching, their mathematics instructional practices, and student achievement (e.g., viewing and discussing the thinking and strategies people have contributed to the solutions).

Many people learned mathematics procedures first and then solved word problems after practising to recite the number facts and carry out number operations by rote. This information was transmitted from the teacher to students. Unlike rote learning, teaching and learning mathematics through problem solving encourages students to reason their way to a solution or a new understanding.

By starting with a problem within a context that activates students’ prior knowledge and mathematical thinking, students can make sense of the problem details and develop solutions
using informal and formal methods. But because each child’s thinking will be different, the classroom instruction must focus on provoking students to relate their informal knowledge to the ideas and strategies developed by a class collectively during a formal lesson. As students examine the work of other students and the teacher, they recognize similarities and differences between and among the solutions and the relationship between informal and formal ways of representing their thinking so they can communicate it.

The communication and reflection that occur during and after the process of problem solving help students not only to articulate and refine their thinking, but also to see the problem they are solving from different perspectives. By seeing a range of strategies that can be used to develop a solution, students can begin to reflect on their own thinking (a process known as “metacognition”) and the thinking of others and to consciously adjust their own strategies in order to improve their solutions.

Through this classroom discourse of questioning, analysis, and argument, students are able to see the concepts expressed from many points of view. Consolidation through classroom discussion renders the mathematical representations explicit and lets students experience different aspects and properties of a mathematics concept, resulting in the students’ deeper understanding.

**Overall Learning Goals for Problem Solving**

Show slide 4. Have the participants analyse the learning goals of this professional learning session by having them identify and explain aspects of the learning goals that are:

- familiar (e.g., components of an effective mathematics classroom – accessibility to manipulatives, students organized in small learning groups . . . I have a list of what should be in a mathematics classroom)
- unfamiliar (e.g., conceptual models of whole numbers and decimals . . . I do not know what “conceptual models” means)
interesting (e.g., role of student-generated strategies for addition and subtraction of whole numbers and decimal numbers . . . I know that student-generated strategies are important, but I don’t know how to get students to share their strategies and how to consolidate them)

**Effective Mathematics Teaching and Learning**

Show slides 5 and 6. Photocopy the two slides, mount them on chart paper, and make a tally chart. See example.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Agree</th>
<th>Disagree</th>
<th>Not Sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics classrooms must be challenging and engaging environments for all students, where students learn significant mathematics.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Students are called to engage in solving rich and relevant problems. These problems offer several entry points so that all students can achieve given sufficient time and support.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lessons are structured so as to build on students’ prior knowledge.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Students develop their own varied solutions to problems and thus develop a deeper understanding of the mathematics involved.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Students consolidate their knowledge through shared and independent practice.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teachers select and/or organize students’ solutions for sharing to highlight the mathematics learning (e.g., bansho, gallery walk, math congress).</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teachers need specific mathematics knowledge and mathematics pedagogy to teach effectively.</td>
<td></td>
<td></td>
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<tr>
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</tr>
</tbody>
</table>

Ask participants to respond to the statements on slides 5 and 6, indicating whether they agree, disagree, or are not sure. Keep a note to yourself of the number of responses to each statement on the chart.
What Does It Mean to Learn Mathematics for Teaching?

Show slide 7. Show Chapter 7, “Doing Mathematics as a Teacher”, of the Deborah Loewenberg Ball webcast, up to and including the analysis of the three different student responses to the multiplication question. If time permits, have the participants explain the mathematics that the students are using in their solutions. Have participants talk about how analysing student work should focus on identifying, discussing, and naming the mathematics inherent in the solution, rather than on finding and marking errors.

Use slides 8 and 9 to add details about the nature of mathematics for teaching. Mathematics for teaching is specialized mathematics knowledge that develops as a result of collaborative and individual teacher engagement in reflection and analysis of teaching and student learning. This reflection and analysis occurs during and after every mathematics lesson taught, each collaborative teacher inquiry/study, any co-teaching experience, and in connection with other implementations of job-embedded professional learning.

Why Study Problem Solving?

Show slides 10 to 12. Ask the participants to examine the expectations for students in Grades 4 to 6 to glean any information that would support the importance of problem solving in a mathematics program. The following are sample responses:

- the high frequency of the mention of problem solving signifies the importance of problem solving as a process for learning mathematics
- problem solving is consistently described across Grades 4, 5, and 6 as an important mathematics learning process
Show slide 13. Highlight the significance of problem solving in the EQAO’s Summary of Results and Strategies for Teachers: Grade 3 and Grade 6 Assessments of Reading, Writing, and Mathematics, 2005–2006. Across the province, problem solving continues to be an area needing improvement at both the Grades 3 and 6 levels.

Show slides 14 and 15. Invite participants to browse through their copies of A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6, Volume 2: Problem Solving and Communication. Provide them with sticky notes. Prompt them to peruse the guide (for about 5 min) to find two ideas about problem solving that are familiar, unfamiliar, and puzzling and mark those pages using sticky notes. Circulate among the participants to take note of what they list as familiar and their queries. Hopefully, during Sessions A, B, and C, their queries will be addressed. The following are possible responses:

- familiar – problem-solving strategies, such as Guess and Test, Make a Diagram
- unfamiliar – characteristics of effective problems
- puzzling – the difference between teaching through problem solving and teaching about problem solving

Learning Goals of the Session

Show slides 16 and 17. Introduce the specific learning goals for Session A. Ask participants to describe what ideas they need to pay attention to during this part of your meeting time. Observation needs to focus on what the teachers are seeing, hearing, and feeling as the students engage in doing their mathematical work.

George Polya’s work on identifying the stages of inquiry in mathematics made him famous as the person who described problem solving as a process. In 1945, he published How to Solve It, which quickly became his most popular publication. It sold over one million copies and has been translated into 17
languages. In his book Polya identifies four basic principles: Understand the problem, Make a plan, Carry out the plan, Look back. The Ontario curriculum incorporates Polya’s problem-solving process. This module focuses on teaching using this process of problem solving.

Finally, through the work teachers do during this session, they will experience ways that questioning and prompts provoke our mathematical thinking.

**Warm Up – Race to Take Up Space**

Show slides 18 and 19. Talk about how this Grade 4 lesson includes a warm up that is designed to activate the mathematical knowledge of Grade 3 students and to see the range of their understanding of area. Have participants use a pair-share strategy to read, analyse, and articulate the expectations for Grades 3 and 4 that they anticipate students will learn.

Show slide 20. To prepare participants for observing students in the classroom, ask them to organize themselves in groups of 3 or 5. Ask each group to assign one person to act as observer while the other participants, working in 2s or 4s, play the game.

Ask observers to start immediately, listening and watching and recording what the mathematics activity sounds like, looks like, and feels like as participants play the game. Observers do not speak, they just write.

Clarify the rules of the game and be sure each group has the required materials. Note that the numbers showing on the faces of the dice are not necessarily the dimensions of the rectangle to be represented. This slide shows a roll of 6 and 4 with a product of 24 – but the rectangle shown includes 24 square tiles in an 8 x 3 arrangement with an area of 24 square units. This is an acceptable response.

After about 10 min, ask 4 or 5 observers to share one idea each from what they recorded. Listen carefully for descriptions of the math actions and prompt
participants until the mathematics is made explicit. Also watch for how children handle materials (e.g., actions with hands using the geoboards and when recording results on square dot paper). When these descriptions are processed, shared learning results.

**Working on It – Carpet Problem**

**Understand the Problem, Make a Plan, and Carry Out the Plan**

Show slide 21. Ask participants to read the problem and, in their groups of 4 or 5, put their heads together and sketch some solutions to the problem. They should discuss what issues they think will arise for students in Grade 4 classes and what tools and materials they would prepare for students to use as they engage in solving the problem. Ask speakers from a few tables to share their insights and anticipation of the mathematics and the problem-solving strategies that will be used for a problem like this.

The major study time during this session is taken in the next section, “Look Back – Reflect and Connect”. It engages teachers in working with a problem-solving vignette from *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6, Volume 2: Problem Solving and Communication*.

**Look Back – Reflect and Connect**

Show slide 22. Explain that the main purpose of having participants read the vignette on pages 18 to 25 of the *Guide* is to focus on strategies used by teachers to prompt student thinking and engagement. The vignette challenges teachers to identify key mathematics and mathematical processes that emerge from the students’ conversations. Teachers can also examine the vignette for ideas about questioning style and prompts that promote the implementation of problem-solving steps that improve student understanding.
Have participants turn to pages 18 to 25 in the Guide and divide the reading among themselves. Ask them to use a reading strategy – reading with a purpose – to answer and discuss reactions to the questions on slides 22 and 23. Allow about 10 min for reading and use the time remaining for discussion.

The following are sample responses:

1. The mathematics:
   - Damon responds to Joel by clarifying that the carpet is not 12 m long — that it has a perimeter of 12 m and that perimeter is the distance around the carpet.
   - One student explains that they need to figure out how much space the carpet will take up. The space the carpet takes up is its area.
   - Students say that most carpets are rectangles and some suggest they could be squares.
   - Students identify how they have successfully used specific manipulatives or grid paper in the past.
   - Students name tools and strategies used by others (e.g., geoboards, square tiles).
   - Students represent their rectangles by saying “. . . is 2 m on the top and bottom and 4 m on the sides.” In this way they identify details about the dimensions of their rectangles and represent the dimensions with the notation 2 x 4.
   - Students state that their rectangle, which is 4 x 2, is similar to the first one that is 2 x 4. Others talk about how they are the same but one is “turned” a different way.

2. The mathematical processes:
   - Students connect to the ideas that carpets are usually rectangular.
   - Students represent the carpet using grid paper, geoboards, and tiles.
   - They select tools to represent the shapes they are thinking about.
   - Other students name the guess and test strategy.
   - Students represent their thinking on paper.
   - Some share their reasoning about how they thought the carpet could be a square, so they divided 12 by 4 to get the length of each of four sides and made a square 3 x 3.
   - They modelled or represented the square carpet with tiles placed in a 3 x 3 array and concluded the area was 9 square units.
   - They reasoned and proceed that the area of the 2 x 4 carpet would be 8 square units.
   - The teacher asks connecting questions about the various shapes and sizes of carpet so students think about carpets they have seen before.
Show slide 23. Listen for responses to questions 4 and 5.

4. Questions the teacher asks to make the problem solving explicit:
   - Do any of you know what materials you will use?
   - Have you seen anyone use another effective strategy?
   - In what ways are the 3 x 3 carpet and the 2 x 4 carpet the same? How are they different? (This highlights the difference between perimeter and area.)

5. Strategies to engage all students:
   - Teacher poses what-if questions to students who are working ahead of others.
   - Teacher asks students to work in pairs.
   - Teacher stops to talk with a few students to summarize where they are and redirect their work by suggesting they write information from the problem on the chart paper.
   - Teacher asks, “Did any of the ideas just shared help you?”

**Next Steps in Our Classroom**

Show slide 24. Ask participants to consider strategies listed on the slide as they work in their classrooms from now until the time you meet again.
Session B – Modelling and Representing Area

Learning Goals of the Session

Show slides 25 and 26. Introduce the specific learning goals for Session B.

What Is Modelling and Representing?

Talk about what it means to model and represent mathematical thinking.

**Modelling** means making an image of your mathematical thinking with concrete materials, sketches, drawings, graphs, words, symbols, or charts.

Models are used to **represent** a mathematical understanding of a situation – the mathematics represented in the problem you are trying to solve. Note that **represent** can also be thought of as re-present. When you move from a concrete image to a drawn or written one, or vice-versa, you are **re-presenting** (i.e., presenting in a different format) your mathematical thinking.

**Re-presenting** an idea in a second or third way allows students to show what mathematics they have linked to the problem situation. Math manipulatives can be used by students because the language of the manipulatives is the universal language of mathematics. If the students can understand the context and relate it to a problem they can imagine, then they can do the mathematics. Then they learn to apply that math knowledge to solve other problems. Once mathematical thinking is described orally, clarified through discussion, and represented, communication is possible.

Enabling students to communicate their mathematical thinking is a major goal of mathematics instruction. We use a great deal of informal mathematical thinking every day. We may not know the formal mathematical representation or the standard algorithms but we can all do mathematics needed to get through daily life. Children also use informal
Mathematical thinking as they live their lives. They learn to solve problems by solving problems but they do not know how that thinking is named and categorized in the discipline of mathematics. Mathematics instruction needs to build on this informal knowledge – to clarify the concepts and build understanding. When connections are made from informal knowledge to the models that represent that knowledge, to the representations that show that knowledge, to the formal representations using the conventions of mathematics (e.g., language and symbols), students are able to communicate about and apply their mathematical understanding to other contexts.

A Guide to Effective Instruction in Mathematics, *Kindergarten to Grade 6: Volume 2*, reports on research about teaching communication. Another teacher inquiry could be launched to engage teachers in examining mathematics communication. This module discusses communication, in the service of learning mathematics through problem solving, but it is a rich enough topic to warrant a study of its own.

**Curriculum Connections**

Show slides 27 and 28. In planning a combined-grades lesson for students, it is necessary to examine the expectations for each grade and understand how they vary. Ask participants to examine the curriculum expectations for Grades 3 to 6 regarding measurement (on BLM 6) and read the parts circled on the slides. Have them record, on sticky notes, what they specifically expect to see and hear from Grade 4 students compared to what they anticipate they will see and hear from Grade 5 students. The following are sample responses:

- both include estimating, measuring, and recording area
- whereas both name “a variety of strategies”, Grade 5 also requires using “a variety of tools”
- Grade 4 expectations focus on understanding of area of rectangles and Grade 5 expectations focus
on understanding of area of regular and irregular polygons

- Grade 5 expectations focus on students creating rectangles and parallelograms with the same areas

The differences in the expectations will mean that when you see the solutions to the lesson problem, you will be looking for different understandings from students in different grades.

**Warm Up – The Size of Things**

This combined-grades class lesson is designed for use in a classroom where students from both grades work together on the warm up, the lesson problem, and “Look Back – Reflect and Connect”.

Prepare **BLM 2A** by copying one sheet for every three or four participants and cutting each sheet up to make small cards. Place each set of cards in an envelope. Note that there are different items listed for work with adults (BLM 2A) and students (BLM 2B).

Show slide 29. Ask participants to clarify the instructions in their small groups. Allow 10 min to 15 min for participants to analyse and discuss their solutions. Ask them to compare their solutions to those of another group.

The problem is quite open ended and questions about the ambiguity of the size of some objects will, no doubt, arise. Participants in each group will define assumptions as they order the objects. Members of a group can define the size of the object they are talking about (e.g., napkin – large dinner cloth napkin or small paper napkin from McDonald’s) and get on with the job of ordering the areas from small to large. The ambiguity of the size of things will come up as an issue when the groups compare solutions to the problem. When two groups disagree on the order (from small to large), the assumptions must be made explicit. The process of clarifying conditions for a problem is essential in the process of doing mathematics and should not be ignored.

Slide 29
Listen for language related to the area of the objects rather than the perimeter or length. Identify the language that you hear and collect words or phrases related to measurement topics. Model the creation of jot notes to track word wall additions. When you gather students together as a math-talk community, make these words explicit. Clarify the meanings of words needed for this lesson (e.g., area, covering space, space inside). Jot down any words you hear that are used improperly and come back to these words as your work on other lessons in the measurement unit progresses.

**Working on It – 4 Square Units Problem**

**Understand the Problem**

Show slide 30. The problem was chosen because it stimulates flexible reasoning about area in terms of square units. Ask participants to talk about their understandings of the words in the lesson problem: polygons, geoboards, and square dot paper.

Polygons are flat, closed shapes with sides that are formed by straight lines that do not cross (intersect).

Geoboards allow for easy exploration of shapes. After users find solutions, the square dot paper encourages them to make permanent records of their thinking. Make explicit the difference between 1 unit length and 1 square unit. Length is measured as the distance between one peg and another horizontally or vertically but not diagonally. To determine length, you count the spaces between the pegs – not the pegs. Area is measured in square units. A polygon with an area of 1 square unit does not have to be in the shape of a square – it can be composed of parts of 1 square unit cut up and rearranged to make another polygon shape.

Square dot paper can be used to represent the pegs on a geoboard and to record a representation of figures created on geoboards.

As you are working through the clarification of these terms, ask participants to set up the materials they
will work with: geoboards, elastics, square dot paper, and pencils.

Show slide 31. The problem introduced in slide 30 has been scaffolded and is shown again on slides 31 and 32. Slide 31 can be used to make sure all participants have the same understanding of what 1 square unit and 2 square units look like on their geoboards and on square dot paper. Talk about the amount of space as the space inside the rubber band – not the distance between pegs.

The second illustration on slide 31 shows a square turned diagonally on a geoboard. Some may say that the area is 1 square unit because they can see one square. The reasoning might be that the rubber band goes from one peg to the next (on the diagonal), so the sides are 1 unit by 1 unit and the square is 1 square unit. To counter this argument, ask participants to compare the size of this square to the size of the 1 square unit that was just created. This “turned” square is bigger than the 1 square unit square. Clearly they cannot both be called 1 square unit of area.

Others may suggest the area is 4 square units because they can draw 4 squares inside the shape. Show the third illustration on the slide and encourage the argument that 4 squares can be drawn with half of each falling inside the diagonal square – so the area must be half + half + half + half or 2 square units. Ask participants to share their reasoning for each polygon they present. All of these words and concepts about measuring on geoboards must be made explicit before the problem solving starts.

**Make a Plan and Carry Out the Plan**

Show slide 32. Ask participants to work with three or four people to solve the 4 Square Units problem and to represent their thinking with each polygon drawn on a separate piece of square dot paper. Also ask participants to label each polygon with the name that refers to the number of sides it has.
**Look Back – Reflect and Connect**

**Constructing a Collective Thinkpad – Bansho as Assessment for Learning**

Show slide 33. In order to make public the mathematical thinking students use to solve a problem, we need a way of organizing the work so everybody can see the range of student thinking. Such an organization allows students to see their own thinking in the context of the similar thinking of others. Students are expected to follow and be able to describe all of the work represented – not just their own. They listen to the explanation of another group and restate, in their own words, the strategies used by the other group. Mathematical ways of talking are modelled and practised – resulting in the creation of a safe, math talk community. All students have a chance to learn more about the mathematics used in developing solutions and to clarify their understanding of the concepts and/or procedures. Through careful management of discourse, the mathematics is made explicit.

Japanese educators call this teaching strategy *bansho*. We will call this process of organizing, displaying, annotating, and discussing solutions *bansho* as well. Bansho engages the teacher in examining student work, organizing it, and displaying it to make explicit the goals of the lesson task. There are sample solutions shown, but be aware that the display is not exhaustive – you may see still more representations. When you do, decide on how they are the same or different from the expected solutions shown on the next page and include them in the discussion.
For example, if the goal was to show as many polygons as possible with an area of 4 square units, the teacher would create an arrangement of student work along the following lines:

• post, on the left of your display wall, squares with areas of 4 square units and bases parallel to the bottom of the page
• post, to the right of that, rectangles with bases parallel to the bottom of the page – solutions that show similar mathematical thinking are arranged in vertical rows that together look like a concrete bar graph
• next, show irregularly shaped figures composed of squares and rectangles
• post, in the fourth column, parallelograms with areas of 4 square units
• next, post triangles with areas of 4 square units
• then, on the right of the display, show other polygons with areas of 4 square units

Your bansho might look like this:
If students can produce such a display, they are definitely demonstrating understanding of the difference between perimeter and area. We are seeing students estimate, measure, and record the area of rectangles (columns one and two) – a Grade 4 expectation. In Grade 5, the actions are required for all polygons, so you want to see figures from the 3rd, 4th, 5th, and 6th columns from your Grade 5 students. Notice that it is the demonstration of the measurement of area (using a variety of tools) that is required and not the calculation of area of all polygons using formulas.

The bansho process uses a visual display of all students’ solutions that is organized from least to most mathematically rich. This is a process of assessment for learning and allows students and teachers to view the full range of mathematical thinking their classmates used to solve the problem. Students have the opportunity to see and to hear many approaches to solving the problem and they are able to consider strategies that connect with the next step in their conceptual understanding of the mathematics. Bansho is NOT about assessment of learning, so there should be no attempt to classify solutions as level 1, level 2, level 3, or level 4.

The matching and comparing conversations focus on the similarities and difference between the displayed mathematics. The teacher makes the learning explicit by using mathematical language to describe and name the concepts and procedures shown in the solutions. Students match their solutions to the displayed ones and examine those of others to learn more about their own thinking. When processes in two solutions match, the second solution is taped above the original to make a bar graph display.

Bansho can also be used to make mathematics explicit during discussion. Annotations could be added to the display that highlight, for example, different names for polygons or show the partial areas that sum to 4 square units.
Refer back to the sticky notes the participants prepared when they were examining the differences in understanding of area expectations for Grades 4 and 5. Use the bansho display to label the demonstrations of the expectations. For this lesson, you want to see if students in Grade 4 can estimate and record the area of rectangles. Grade 5 students need to estimate and measure the area of regular and irregular polygons and create shapes with a specified area. The other expectations will need to be addressed in future lessons in this same unit of study.

Show slide 34. Facilitate the conversation about the readings on pages 32–34 of *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6: Volume 2*.

The following are sample responses:

1. a. The purpose of questioning and prompting students during problem solving is:
   i. to help students engage and predict;
   and/or
   ii. to help them make connections to the context and understand the problem.

   b. The purpose of questioning and prompting students after problem solving is:
   i. to get students to share their understanding of the mathematics they used to solve the problem;
   ii. to help students retell and share strategies;
   and/or
   iii. to make the representations clear so others can see the mathematics in the solutions.

2. When preparing questions for reflecting and connecting, teachers need to remember to:
   a. ask questions about understanding rather than factual recall;
   b. avoid yes-no questions; and
   c. avoid leading questions.
3. Think-alouds can be used to model any stage of the problem-solving process. Teachers verbalize the thinking and decision making that they use or that other students have used to engage in the problem. Think-alouds allow teachers to demonstrate reflecting at each stage.

Questioning and prompting during reflecting and connecting allows students to practise listening and speaking mathematical language that explains their solutions. After listening to one participant explain his or her solution, ask other participants to say what the first person said – to retell what she or he understands about the solution. Ideally, the conversation during bancho and reflecting and connecting moves between teacher to student to student to teacher to student to student to student again. This reinforces the importance of students listening to each other and adapting their thoughts as they learn from one another. The classroom becomes a knowledge-building environment. Each person leaves the learning with more than he or she brought to the lesson. The teacher’s job is to get all of the mathematical ideas on the table during the lesson and to sort and classify the language so the big ideas of the lesson are made explicit for all by the end of learning.

**Next Steps in Our Classroom**

Show slide 35. Ask participants to set the goals listed on the slide for their classroom work between now and your next session together. Ask them to be prepared to share news of their experiences and the work that students produced.
Session C – Organizing and Coordinating Student Solutions to Problems Using Criteria

Learning Goals of the Session

Show slides 36 and 37. Have participants read the specific learning goals of Session C and identify the mathematics teaching strategies that they will experience during this session. Use a think-pair-share strategy. The following are possible responses:

- recognizing students’ mathematical thinking in their solutions
- understanding the mathematical relationships between students’ different solutions to a problem
- organizing student solutions in a class display for student learning
- coordinating student discussion of solutions to develop mathematical thinking and communication

Curriculum Connections

Show slides 38 and 39. The purpose here is to highlight the learning goals of the problems in this session. To activate participants’ knowledge of measurement expectations, ask them to identify and synthesize in one or two sentences the mathematics highlighted in the curriculum expectations. Use a round robin strategy (i.e., each person takes turns sharing their synthesis), so that participants are building on each other’s ideas to develop a precise synthesis statement of the curriculum expectations for this problem. Some possible responses are:

- Grade 5 – estimate, measure, record, and calculate the area of rectangles
- Grade 5 – describe the relationship between the side lengths and the area of rectangles
- Grade 6 – estimate, measure, record, and calculate the area of triangles
- Grade 6 – describe the relationship between the area of rectangles and triangles
Warm Up – Composite Shape Problem

Show slide 40. This warm-up problem is useful as it provides students with practice in composing shapes to create a different composite shape. Students visualize the decomposing of a shape into rectangles (Grade 5) and triangles (Grade 6) and activate the thinking process of composing shapes differently to create composite shapes (L-shaped polygons or irregular hexagons).

When teachers work on this warm up, they may recognize it as an extension of the previous lesson problem in Session B. Because your participants will be familiar with this problem, it should require less time to solve. It does not require participants to calculate the area of rectangles, parallelograms, and triangles, but it does require them to conceptualize the measuring of area of the shapes by counting the number of square units of space the shape covers or takes up.

To activate participants’ thinking-through of the criteria for this problem, do a think-aloud to highlight the sense making of the problem and the development of the solution.

• A Grade 5 example: What could a rectangle look like that has an area of 4 square units? What could a rectangle of 4 square units look like if I decomposed it into 3 rectangles or rearranged the 3 rectangles into another polygon

• A Grade 6 example: What could a rectangle look like that has an area of 4 square units? What could a rectangle of 4 square units look like if I decomposed it so that it has one or more rectangles, parallelograms, and triangles? Could I rearrange the decomposed shapes to form another polygon with a total area of 4 square units?
Some possible strategies are:

- Start with a familiar shape like a rectangle that is 4 square units and decompose it into 3 rectangles, or a combination of rectangle, parallelogram, and triangle, then rearrange the decomposed shapes to make a different composite shape with an area of 4 square units.

- Construct 3 rectangles that have a total area of 4 square units and compose them into a composite shape.

- Construct a combination of rectangles, triangles, and parallelograms that have a total of 4 square units and compose them into a composite shape.

To justify the accuracy of the composite shape, ask participants to refer back to the problem details, which include the criteria of the composite shapes. Reflecting on the problem details is an important strategy to check whether a solution created is actually a relevant solution to the problem.

**Working on It – L-shaped Problem**

Show slide 41. This problem is a useful one for a combined Grades 5 and 6 classroom, as it is structured to have differentiated prompts that address Grades 5 and 6 curriculum expectations, using the same problem. This is another approach to differentiating curriculum for combined grades – different from the approach in Session B. The prompts are different for Grades 5 (rectangles) and 6 (triangles) students.

**Understand the Problem**

Have participants discuss the key ideas that they will use to make a plan to solve the problem. Use a pair-share strategy. Ask participants to identify the information that they will use to make their plan (e.g., linear dimensions of the L-shape [irregular hexagon], the relationship between the L-shape and other known shapes, like rectangles and triangles). Discuss the importance of making the key information about the problem public to all students. All mathematical thinking and strategies used to solve the
problem should be made explicit in the classroom, so all students have access to problem solving. They will learn by listening to one another.

To develop a Grade 5 perspective that builds from prior knowledge and experience, ask participants to explain how measuring the area of this L-shaped figure or irregular hexagon (6-sided shape) is different from measuring the area of a rectangle. The following are possible responses:

- the rectangle has 4 sides and the hexagon has 6 sides
- the irregular hexagon looks like it is composed of 2 rectangles, so it is like measuring the area of 2 rectangles

To develop a Grade 6 perspective that builds from prior knowledge and experience, ask participants to explain how measuring the area of this L-shaped or irregular hexagon is similar to measuring the area of a composite shape made up of rectangles, parallelograms, and triangles. The following are possible responses:

- the irregular hexagon looks like it is composed of 2 rectangles that can be decomposed into 4 triangles if I draw a diagonal line from opposite vertices of the rectangle
- the irregular hexagon can be recomposed to make a different shape, like a rectangle

**Make a Plan and Carry Out the Plan**

Provide participants with square grid chart paper, square tiles, and/or base ten blocks to represent their thinking when solving the L-shaped problem. Ask participants to identify the information from the problem that they will use to make their plan. Ask participants to explain what it means to estimate, measure, and calculate the area of the L-shaped figure (irregular hexagon). The following are possible responses:

- estimate – overlay a triangular grid or square grid over the L-shape and count the number of square units the shape covers
- measure – overlay a square grid that is 1 cm x 1 cm per each square and count the number of square units the shape covers; decompose the L-shaped figure into different-sized rectangles and determine the area of part A and the area of part B by multiplying the number of rows of squares by the number of columns of squares in each rectangle (i.e., rectangle A has 8 rows of square units and 4 columns of square units – 8 rows x 4 columns – 32 square units)
- calculate – decompose the area of the L-shaped figure into rectangles and calculate the area of the rectangles by multiplying the length of the sides of the rectangle (i.e., rectangle A has linear dimensions of 8 linear units x 4 linear units = 32 square units)
Have participants develop solutions to the L-shaped problem in pairs or in small groups using square grid paper and manipulatives. Listen and watch for the mathematics they are using to develop solutions. Use the math vocabulary (arrays, columns, compose, decompose, linear dimensions, rows, square units) to name the mathematics they are using in their solutions. As you circulate among participants, think about how you will sort and classify their solutions to the L-shaped problem using criteria such as:

- strategies for measuring and/or calculating the area (e.g., counting square units, measuring area by counting the number of square units in arrays, calculating area by multiplying the linear dimensions of rectangles decomposed from the L-shaped figure);
- measuring area or calculating area;
- number of partitions made to measure the area of rectangles within the L-shaped figure; and/or
- part-to-whole and whole-to-part representations.

**Constructing a Collective Thinkpad – Bansho as Assessment for Learning**

Show slides 42 to 44. Prompt participants to respond to questions 1 and 2 by referring to pages 48–50 in A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6: Volume 2, Problem Solving and Communication. The following are possible responses:

1. What does the teacher need to do to understand the range of student responses?
   - recognize the student’s ability to take existing information into a new situation and know how to use it
   - know that the teacher can recognize solutions that make sense and those that are reasonable
   - understand that some problems can be interpreted in more than one way
   - provide various students with opportunities to share their strategies for solving the problem in different ways
• provide prompts that help to scaffold and support students' developing ideas
• ask students questions to develop their strategies for reasoning their way towards an answer
• celebrate diversity in thinking

2. What does the teacher need to know and do to coordinate class discussion so it builds on the mathematical knowledge from student responses?
• build a strategy wall as students in the class generate new strategies
• provide encouragement to students as they begin to make conceptual connections among the different solutions
• promote students' openness to the ideas of others
• encourage students' willingness to try new ways or strategies

Recording a Range of Student Thinking – Assessment for Learning

As you circulate among participants, record different solutions to the problem that you anticipate should be shared during the whole class discussion of the solutions. Use this assessment for learning tool to record your observations of student work and interviews with students as they are developing their solutions.

This assessment for learning recording tool is intended to be a one-page snapshot of the mathematics that the students are using during a problem-solving lesson. See BLM 4.

The key steps are:

A Write the title of the mathematics unit of study.
B Write the mathematics problem.
C Write the learning goal of the lesson and the grade-specific curriculum expectations.
D Represent different solutions to the problem developed by the teacher prior to the lesson and other solutions that the students develop during the lesson. Number each solution so that they can be referenced to each student’s solution.
E Record the names of the students in each cell, according to their classroom seating. The solutions recorded that are similar to the students’ solutions can be written using the number code described in D.
F Record student errors noticed in the solutions, so that key misconceptions can be addressed during class discussion or followed up with in small group interventions.
G Record any questions that you have about students’ learning to investigate with other teachers or in resource materials.
Other assessment strategies and recording tools are described on pages 48–51 of *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6: Volume 2, Problem Solving and Communication*.

**Organizing to See a Range of Student Thinking – Bansho**

Organize the different solutions on a blackboard or whiteboard space. Label the classifications according to the criteria used for organizing the solutions. Annotate the solutions using the mathematics vocabulary. Two possible examples of bansho for the L-shaped problem are provided below.

**Grade 5 Bansho of Solutions to L-Shaped Problem Using Rectangles**

![Grade 5 Bansho of Solutions to L-Shaped Problem Using Rectangles](image)

**Grade 6 Bansho of Solutions to L-Shaped Problem Using Triangles**

![Grade 6 Bansho of Solutions to L-Shaped Problem Using Triangles](image)
To make sense of different student solutions to a problem, the teacher needs to understand the mathematics inherent in each of the student solutions, as well as the mathematical connections among the solutions. For example, in the Grade 5 bansho, the last two solutions represent the idea of creating a larger rectangle and decomposing its parts to determine the area of the L-shape, while the other seven solutions show how the rectangles are partitioned to compose the whole. Yet all of the solutions could include the idea of measuring the area of rectangles by multiplying the number of rows and columns of square units or calculating the area by multiplying the linear dimensions of each rectangle that compose the L-shaped figure (irregular hexagon). Some key mathematical ideas inherent in these solutions are as follows:

- area can be estimated and measured using square units
- to avoid counting each square unit individually, an array of square units can be organized so that the number of rows and columns can be multiplied to give the area of the rectangle in square units
- the linear dimensions of a rectangle can be multiplied to give the area of the rectangle in square units

Discussion prompts to extend students thinking include:

Will these strategies work for any size L-shaped problem? The following is a possible response:

- all the strategies will work except for the last one where the L-shaped figure is rotated to create a larger rectangle – because one part of the L-shape is shorter than half the length of the other, when you rotate it, a space is created in the middle of the rectangle

What's the relationship between calculating the area of a rectangle and calculating the area of a triangle? The following are possible responses:

- a triangle can be created from a rectangle by dividing it in half through a diagonal
- if I draw a triangle, I can divide the triangle in half and recombine the two triangles to make a rectangle
Look Back – Reflect and Connect

Show slides 45 to 47. Have participants refer to the Ontario Curriculum, Grades 1 to 8: Mathematics, 2005 and A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6, Volume 2: Problem Solving and Communication, pages 38–44. Divide the participants into four groups, each of which will focus on one of the four debriefing questions.

1. What mathematics is evident in the solutions?

The following are possible responses:

- decomposed the L-shaped figure into rectangles or triangles
- composed rectangles or triangles to make an L-shape
- counted the number of square units that the L-shape covers
- organized the square units for each rectangle in an array of rows and columns of square units
- related the linear dimensions of a rectangle to its area through multiplication of the two linear dimensions

2. Which problem-solving strategies were used to develop solutions?

The following are possible responses:

- Make a Model – created models of the L-shaped figure using each square tile or unit block to represent one square unit of the L-shaped figure (e.g., the concrete materials used were base ten blocks and colour square tiles)
- Draw a Diagram – used the square grid paper to represent the L-shaped figure and its dimensions, so different decompositions or compositions of the L-shaped figure could be carried out
- Solve a Similar Problem – for the triangle responses, decomposed the L-shape into rectangles because it is similar to decomposing the L-shape into triangles
• Use or Find a Pattern – when the L-shaped figure was decomposed into 2 rectangles through a horizontal partition, it was logical to decompose it into 2 rectangles, through a vertical partition, then continue to decompose the shape, etc.

3. How are the mathematical processes evident in the development of the solutions?
   The following are possible responses:
   a) Problem Solving – selected and applied problem-solving strategies (e.g., made a model, drew a diagram, solved a similar problem, used or found a pattern)
   b) Reasoning and Proving – saw the mathematical relationships between the different solutions to the L-shaped problem as displayed on the bansho, such as the horizontal and vertical partitions of the L-shaped figure into rectangles or triangles
   c) Reflecting – explained their area solution in relation to the measuring and calculation of area of rectangles and triangles
   d) Selecting Tools and Computation Strategies – used visual tool (square grid paper drawings) and concrete materials (base ten blocks, colour square tiles)
   e) Connecting – considered the area of separate sections in their classroom (e.g., the entry to classroom may cause a smaller rectangle for one part of the room and a larger one for the rest of the room, a living/dining room area may share walls)
   f) Representing – showed solutions as a set of diagrams, or created one concrete model that had 40 square units and manipulated the grouping of the square units to show the area of the decomposed rectangles
   g) Communicating – communicated mathematical thinking orally (e.g., explained their solution to others in the class), visually (e.g., used drawings and concrete models), and in writing (e.g., explained ideas in words with a diagram)

4. What are some ways that the teacher can support student problem solving?
   The following are sample responses:
   • Organize mathematics lesson as a three-part lesson: Warm Up, Working on It, Look Back – Reflect and Connect
   • Encourage students to make connections with their prior knowledge
   • Answer students’ questions, but avoid providing a solution to the problem
   • Expect students to defend their procedures and justify their answers
   • Use a variety of concrete, pictorial, and numerical representations to demonstrate a problem solution
   • Persevere in the problem-solving process
   • Step back and resist the impulse to give students the answer or be overly prescriptive about how the problem should be solved
   • Use wait time
   • Avoid yes-no, leading, and teacher-centred questions
- Organize class discussion so students share their representations of mathematical situations
- Prompt students to reflect on their mathematical work

**Next Steps in Our Classroom**

Show slide 48. The tasks are designed so that teachers can apply and practise their learning from Session C in their classroom. For example, participants are asked to:

- identify the mathematics evident in 4 student mathematics work samples;
- identify the problem-solving strategies evident in 4 student mathematics work samples; and

Participants should be prepared to bring the results of their classroom application of the ideas and strategies from Session C to Session D. Provide time at the start of the next session for participants to share their classroom experiences in applying the new ideas and strategies from Session C.
Session D – Selecting and Writing Effective Mathematics Problems for Learning

Learning Goals of the Session

Show slides 49 and 50. Have participants read the specific learning goals of Session D and identify the strategies that they will experience during this session. Use a think-pair-share strategy.

The following are possible responses:

- understanding the purpose of using problems for learning mathematics
- being familiar with the characteristics of effective problems
- analysing problems from resource materials to determine the effectiveness of problems
- selecting, adapting, and/or writing problems

Warm Up – About Problems

Show slides 51 and 52. Divide participants into four groups, each of which will focus on either questions 1 and 2 or questions 3 and 4. Give a copy of BLM 5 T-Charts for Comparing Ideas to each participant. They will be using the Guide to Effective Instruction, Kindergarten to Grade 6 (Vol. 2, pp. 6–7 and 26–28) to respond to questions 2 and 4. Some possible responses are provided below.
1. What do you think are the purposes of problems in terms of learning mathematics?
• to have students make connections between mathematics and real world contexts
• to use higher order thinking skills
• to learn mathematics
• to learn how to apply knowledge in practical settings

2. How are the ideas about problems on pages 6 and 7 similar to and different from your ideas?
• to explore, develop, and apply conceptual understanding of a mathematical concept (teaching through problem solving)
• to develop inquiry or problem-solving processes and strategies (teaching about problem solving)

3. What do you think are the key aspects of effective mathematics problems?
• have a real-life context
• require gathering, organizing, and analysing information to make a decision
• the answer isn’t immediately obvious
• provokes the problem-solving process: Understand the problem, Make a plan, Carry out the plan, Look back, and Communicate

4. How are the ideas about mathematics problems on pages 26 to 28 similar to and different from your ideas?
• more than one solution
• promotes the use of one or more strategies
• provides a learning situation related to a key concept or big idea
• solution is not immediately obvious
• context is meaningful to students
• solution time is reasonable

Working on It – Analysis of Problems
Show slide 53. Ask the participants why it might be important to analyse problems from different resource materials (e.g., textbook programs, resource books, activity books) before using them for teaching mathematics. The following are possible responses:
• to ensure they are aligned to grade-specific mathematics curriculum
• to determine if the problem will promote more than one solution
• to decide if the problem is useful for learning a particular mathematics concept

Understand the Problem
Divide participants into six groups, and have each group analyse one of the six problems from Sessions A, B, and C. Direct the participants to use the Criteria for Effective Mathematics Problems from slide 53.
Have each group share its analysis of the assigned problem with the whole group. Prompt participants to provide specific details about the problem that match the criteria. Note that each problem does meet all of the Criteria for Effective Mathematics Problems.

Show slides 54, 55, and 56. Have participants discuss the relationships among the six problems, using a pair-share strategy. To prompt their thinking, show the Curriculum Connections in slides 54 to 56.

The following are possible responses:

- in terms of the curriculum expectations, the 6 problems show the conceptual development of measuring and calculating the area of polygons; that is, from Grade 3 the focus is on conceptually measuring area using square units while in Grade 5, the focus is on calculating the area of rectangles, and in Grade 6, the focus is on calculating the area of triangles

- the 6 problems show Before and During part of a 3-part lesson (e.g., problems a, c, and e are warm-up problems to activate prior knowledge, and problems b, d, and f are lesson problems for learning mathematics)

- the 6 problems focus on the big idea of part-whole relationships, that is, composing parts to make a composite shape and decomposing a shape into other shapes

If participants require more practice in discerning the quality of problems using the Criteria for Effective Mathematics Problems, have them analyse the initial and revised problems on page 28 of *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6, Volume 2: Problem Solving and Communication*. Pose questions like, What is the difference between the quality of the “initial problems” and the “revised problems”? and How do the revised problems compare to the Criteria for Effective Mathematics Problems?
Make a Plan

Show slide 57. This slide summarizes the problems experienced during Sessions A, B, and C and shows where they fit in terms of a three-part lesson. The three-part lesson outlines the flow of a problem-solving-based lesson.

The first part is an introduction (Warm Up) that activates students’ prior mathematics knowledge and experiences that are relevant and can be built on in relation to the lesson learning goal. The problems, the Race to Take Up Space, the Size of Things, and the Composite Shape Problem, are examples of warm-up problems.

The second part is the teaching/learning (Working on It) part of the three-part lesson that provokes new mathematical thinking in relation to prior knowledge and experiences. The problem should meet the Criteria for Effective Mathematics Problems and be focused on students learning grade-specific mathematics expectations, as per the Ontario Curriculum, Grades 1 to 8: Mathematics, 2005. The problems: The Carpet Problem, 4 Square Units Problem, and the L-Shaped Problem are examples of teaching/learning problems.

Show slide 58. The third part is the consolidation part of the three-part lesson. Consolidation includes looking back or reflecting on the learning gained from solving the lesson problem and practising the new learning by solving additional problems focused on the lesson learning goal. During consolidation, students need to practise on their own and in pairs and receive feedback from their peers and teacher about their solutions and strategies. Careful selection of consolidation problems ensures that students practise what they just experienced and learned during the lesson problem.
**Carry Out the Plan**

Organize participants so that one group is focused on writing one problem. Have participants identify the Criteria for Effective Mathematics Problems that they will begin to use to write consolidation problems for Sessions A, B, and C based on the Working-on-It problems. For example, focusing on the curriculum expectations should be the starting point for writing effective problems. Provide participants with BLM 6 – Grades 3 to 6 Curriculum Connections for Area – to use as a starting point for their writing of consolidation problems.

In addition to the Criteria for Effective Mathematics Problems, also consider the structure of the problems, in terms of readability and comprehension. EQAO assessment questions are good examples of problems that provide a consistent structure that intentionally supports the reader in understanding the problem.

Consider these features of text:

<table>
<thead>
<tr>
<th><strong>Conceptual Readability of Text</strong></th>
<th><strong>Design Factors for Readability of Text</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>• organization of text lines – start with a context statement, state the problem/question, math action to take, materials to use, and type of detail needed in solutions (See problem #2 below.)</td>
<td>• diagrams and text – use diagrams and colour carefully and intentionally alongside text to illuminate the content</td>
</tr>
<tr>
<td>• sequence of words – avoid phrases and put the math action at the start of the sentence</td>
<td>• amount of text – use as few words as possible</td>
</tr>
<tr>
<td>• density of concepts, abstractness of ideas – use the same nouns and verbs</td>
<td>• sentence length – use as few words as possible</td>
</tr>
<tr>
<td>• coherence of ideas – check if all the prompts in parts a, b, and c relate to the central question/problem</td>
<td>• sentence structure – be consistent in providing the question prior to the statements of math action</td>
</tr>
<tr>
<td>• variability of vocabulary – avoid using different prompts for the same math action (e.g., if you want students to explain their thinking, say “Explain your thinking,” rather than using a different prompt that is intended to have the same meaning)</td>
<td>• line length – avoid sentences that run for more than two lines, as it is difficult for a struggling reader to track</td>
</tr>
<tr>
<td></td>
<td>• text placement – as it relates to diagrams and matching key words</td>
</tr>
<tr>
<td></td>
<td>• punctuation – use to break up text</td>
</tr>
<tr>
<td></td>
<td>• text form or genre – use narrative text to provide context and use expository text for instructions</td>
</tr>
</tbody>
</table>
For example, here is a scaffolded problem that is difficult for learners to read:

1. Here are two arrangements of chairs in a meeting room. How can I tell if both arrangements have the same number of chairs? Describe each rectangle with a multiplication sentence. How are the first factors related? The second factors? Make the first rectangle. Then move two rows to make the second one. How does this explain the relationships between the factors?

An improved version of the scaffolded problem could be:

2. Here are two arrangements of chairs in a meeting room. How can I tell if both arrangements have the same number of chairs?
   a) Write the multiplication sentence for each rectangle.
   b) List the factors for each rectangle.
   c) How are the factors related to one another?
   d) Explain your ideas using counters.

Some sample consolidation problems are as follows:

- Session A – If the length of the fence around the primary playground is 80 m, what could the area of the playground be? Explain your thinking, using diagrams on square grid paper.
- Session B – Is it possible to cover 10 square units using only these shapes: rectangle, two triangles, three squares, and four parallelograms? Justify your answer, using diagrams on square grid paper.
- Session C – Construct two different L-shaped figures. Each figure should have an area of 40 square units. Show your area calculations.

Have some participants share their problems and prompt other participants to analyse the problems shared using the Criteria for Effective Mathematics Problems. Briefly provide discussion time for feedback from participants for improving the content and structure of the problems.
Look Back – Reflect and Connect

Show slide 59. It is important that participants solve one another’s consolidation problems, so that they can be aware of the mathematics that the problems are provoking them to use. Such experience will provide participants with the understanding that effective mathematics problems provoke particular mathematical thinking that is focused on intended learning goals. Also, in solving each other’s problems, participants will gain insight into difficulties that students have in interpreting the intent of the problem as well the cognitive pathways that problems may require students to take. Scaffolding prompts in a problem assist students along a pathway of thinking, without telling them what to think.

Circulate among participants to read their problems and to analyse the mathematical solutions developed. Prompt participants to be explicit and detailed about the mathematics they are using in their solutions. For teachers to be able to understand the range of students’ mathematical thinking in solutions, they need to develop precise mathematical language and discern details in solutions. The ability to understand and evaluate the mathematical significance of students’ solutions is a key aspect of mathematics for teaching. As well, such mathematics knowledge and skill aids the teacher in coordinating students’ comments and questions as solutions to the lesson problem are discussed.

Show slide 60. Have one person in each group read one of pages 30 to 34 from the Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6, Volume 2. Readers are looking for ideas to answer the question, What are some ways that the teacher should support student problem solving for these consolidation problems? When participants seem to have finished reading their one page, prompt them to take turns sharing their ideas from the reading, using a round robin cooperative learning strategy (participants take turns sharing their ideas).
The following are possible responses:

- bring students back together to share and analyse solutions
- highlight the big ideas and key concepts
- encourage the use of a variety of concrete, pictorial, and numerical representations to demonstrate a problem solution
- step back and resist the impulse to give students the answer or to be overly prescriptive in how the problem should be solved

**Next Steps in Our Classroom**

Show slide 61. During the implementation of your job-embedded professional learning process – studying the materials in Sessions A, B, C, and D – suggestions are provided for tasks to support learning and to engage teachers back in their classrooms and principals in their schools. The tasks are designed so that teachers can apply and practise their learning in their classrooms. For example, after Session D, participants are asked to:

- compare the problems from their resource materials (e.g., textbook programs, teacher resource books) to the Criteria for Effective Mathematics Problems
- gather student solutions to a resource material consolidation problem
- write consolidation problems that improve on current consolidation problems in resource materials
- gather student solutions to the new consolidation problems
- compare student solutions to old consolidation problems with those for new ones

If possible, have teachers share their consolidation problems using inter-school mail or email, so that they can see one another’s application of improving problems using the Criteria for Effective Mathematics Problems. This kind of work also builds their resources of grade-specific, effective mathematics problems.
Show slide 62. To conclude the learning from Sessions A, B, C, and D, provide a few moments for participants to reflect on their learning and achievement of the overall learning goals of this professional learning module on problem solving.

Show slide 63. Use this slide to promote the importance of ongoing professional learning goals. These five goals are central to learning *mathematics for teaching* and should be considered an overall goal for any mathematics professional learning session. Challenge your participants to take up at least one of these goals, as a focus for their daily professional learning, as they plan and carry out mathematics lessons in their classrooms.

**Professional Learning Opportunities**

Show slides 64 and 65. Share the many ways in which Ontario teachers continue their professional development. Ask, What are the next steps in your school? In your board?
References


Resources to Investigate

Coaching Institute for Literacy and Numeracy Leaders, video on demand, available at www.curriculum.org.
Black Line Masters

• BLM 1, Square Tile Paper (1”)
• BLM 2A, Size-It-Up Cards for Adults
• BLM 2B, Size-It-Up Cards for Students
• BLM 3, Square Dot Paper
• BLM 4, Assessment for Learning Seating Plan Tool
• BLM 5, T-Charts for Comparing Ideas
• BLM 6, Grades 3 to 6 Curriculum Expectations for Area
BLM 1 – Square Tile Paper (1")
<table>
<thead>
<tr>
<th>a small sticky note</th>
<th>a cheque</th>
</tr>
</thead>
<tbody>
<tr>
<td>a five dollar bill</td>
<td>a business card</td>
</tr>
<tr>
<td>a credit card</td>
<td>a cocktail napkin</td>
</tr>
<tr>
<td>a CD case</td>
<td>a 30 cm ruler</td>
</tr>
<tr>
<td>a paperback cover</td>
<td>an envelope</td>
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</tbody>
</table>
### BLM 2B – Size-It-Up Cards for Students

<table>
<thead>
<tr>
<th>an item</th>
<th>another item</th>
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</thead>
<tbody>
<tr>
<td>a hockey card</td>
<td>a computer screen</td>
</tr>
<tr>
<td>a math textbook</td>
<td>a five dollar bill</td>
</tr>
<tr>
<td>a playing card</td>
<td>an envelope</td>
</tr>
<tr>
<td>a napkin</td>
<td>a CD case</td>
</tr>
<tr>
<td>a ruler</td>
<td>a report card</td>
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</tbody>
</table>
BLM 3 – Square Dot Paper

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# BLM 4 – Assessment for Learning Seating Plan Tool

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Grade</th>
<th>Assessment For Learning Observation &amp; Interview</th>
<th>Term</th>
<th>Year</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Unit of Study</th>
<th>Date</th>
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</thead>
</table>

<table>
<thead>
<tr>
<th>Mathematics Lesson Task/Problem</th>
<th>Learning Goal/Curriculum Expectations</th>
<th>Possible Solutions</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Questions about student learning...</th>
</tr>
</thead>
</table>

| Student Errors |
BLM 5 – T-Charts for Comparing Ideas

<table>
<thead>
<tr>
<th>1. What do you think are the purposes of problems in terms of learning mathematics?</th>
<th>2. How are the ideas about problems on pages 6 and 7 similar to and different from your ideas?</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. What do you think are the key aspects of effective mathematics problems?</td>
<td>4. How are the ideas about mathematics problems on pages 26 to 28 similar to and different from your ideas?</td>
</tr>
</tbody>
</table>
BLM 6 – Grades 3 to 6 Curriculum Connections for Area

Session A
• 3m37 - Estimate, measure (i.e., using centimeter grid paper, arrays), and record area (e.g., if a row of 10 connecting cubes is approximately the width of a book, skip counting down the cover of the book with the row of cubes [i.e., counting 10, 20, 30, ...] is one way to determine the area of the book cover).
• 4m50 - Determine, through investigation, the relationship between the side lengths of a rectangle and its perimeter and area. (Sample problem: Create a variety of rectangles on a geoboard. Record the length, width, area, and perimeter of each rectangle on a chart. Identify relationships.).
• 4m51 - Pose and solve meaningful problems that require the ability to distinguish perimeter and area.

Session B
• 3m29 - Estimate, measure, and record area using standard units.
• 3m44 - Describe through investigation using grid paper, the relationship between the size of a unit of area and the number of units needed to cover a surface.
• 4m38 - Estimate, measure, and record area, using a variety of strategies.
• 4m39 - Determine the relationships among units and measurable attributes, including the area and perimeter of rectangles.
• 4m51 - Pose and solve meaningful problems that require the ability to distinguish perimeter and area.
• 5m31 - Estimate, measure, and record area using a variety of strategies.
• 5m36 - Estimate and measure the perimeter and area of regular and irregular polygons.
• 5m40 – Create, through investigation using a variety of tools and strategies, two-dimensional shapes with the same area.
• 6m35 - Construct a rectangle, a square, a triangle, and a parallelogram using a variety of tools given the area.

Session C
• 5m36 - Estimate and measure the area of regular and irregular polygons using a variety of tools.
• 5m41 – Determine, through investigation using a variety of tools and strategies, the relationships between the length and width of a rectangle and its area and generalize to develop a formula.
• 6m35 - Construct a rectangle, a square, a triangle using a variety of tools.
• 6m36 – Determine, through investigation using a variety of tools and strategies, the relationship between the area of rectangle and the area of triangle by decomposing and composing.
• 6m42 - Solve problems involving the estimation and calculation of the area of triangles.