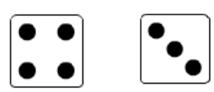
Developing Computational Proficiency with Addition and Subtraction

Sequence for Teaching Addition Facts and Strategies

Although there is no one correct way to sequence the teaching of addition and subtraction facts, the facts students find easiest to master can be considered as foundational. More challenging facts can be derived from the known.

> Double combinations, such as 2 + 2 or 5 + 5, seem to be the easiest to recall, followed by combinations of plus 1 or plus 2.
>
> *(Kamii 2000, Moomaw 2011)*

Studies have shown a highly significant difference between children's ability to add with concrete representations as opposed to only numerals. NCTM shares, in its August 2015, Vol. 22, Issue 1, details of students being much more successful adding cards with dots than cards with only numerals. The imposition of numerals masked their conceptual understanding.



"Four and three is …"

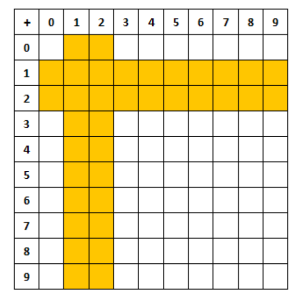
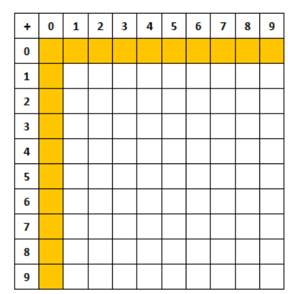
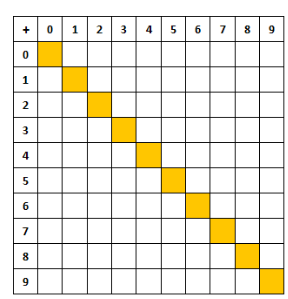
The Ontario Grade 1 mathematics curriculum supports the need for concrete representations. By the end of Grade 1, the curriculum calls for children to solve a variety of problems involving addition and subtraction of whole numbers to 20, using concrete materials and drawings.

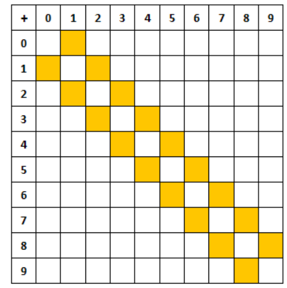
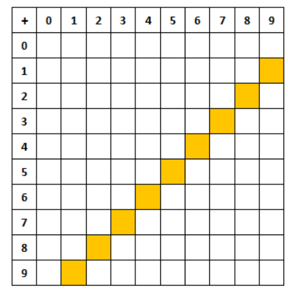
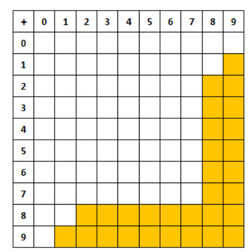
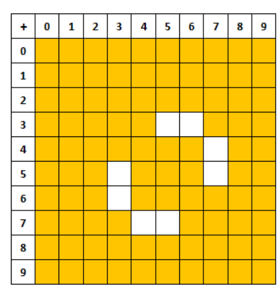
> In today's math classroom, we want children to do more than just memorize math facts. We want them to understand the math facts they are being asked to memorize. Our goal is automaticity and understanding; without both, our children will never build the foundational skills needed to do more complex math.
>
> *Susan O'Connell, coauthor of Putting the Practices Into Action, Mastering the Basic Math Facts in Addition and Subtraction, and Mastering the Basic Math Facts in Multiplication and Division*

Ways to sequence addition and subtraction facts and computational strategies with consideration of the need for concrete representations are outlined on pages 2 and 3.

Addition: Establishing the Foundational Facts and Strategies

In order for students to develop efficient addition strategies, teachers need to have at their command as many strategies as possible, even if they do not use these strategies themselves. When teachers pose good problems, then recognize students' invented strategies, they can capitalize on students' ideas and establish classroom language for identifying and practising sets of facts and corresponding strategies. The following sets of facts and strategies show one-digit numbers but apply to multi-digit

Sets of Facts and Strategies	Examples and Visual Summary
One-More-Than and Two-More-Than Facts	Examples: 1 + 5, 7 + 2
Ordering problems lay the groundwork for the One-More-Than facts, even though addition will not be mentioned at the time when the following question is posed: "What number is 1 more than 4?" By labeling this type of fact retrieval "One-More-Than", teachers help students identify sets of addition facts that involve the addition of 1. The strategy of naming the next counting number is a quick way to answer the question, "What is 1 more than __?" Many children will be ready to extend this idea to Two-More-Than facts, (e.g., What number is 2 more than 7?). Some may extend the idea to subtraction contexts worded as "Six is two more than what number?"	
Facts with Zero	Examples: 0 + 3, 7 + 0
These problems are generally easy for students. However, many students over-generalize the idea that addition increases a value and subtraction decreases a value, and this is a chance to clarify the misconceptions. When solving problems involving zero, you can illustrate an empty part of a Part-Part-Whole relationship, or model addition or subtraction of nothing.	
Doubles	Examples: 2 + 2, 8 + 8
"There are only ten doubles facts from 0 + 0 to 9 + 9 as shown here. These ten facts are relatively easy for most children to learn and become a powerful way to learn the near-doubles (addends one apart)." John A. Van de Walle, 2001 (Elementary and Middle School Mathematics: Teaching Developmentally, 4th Edition) Modelling two identical sets and counting by twos connects to skip-counting.	

Sets of Facts and Strategies	Examples and Visual Summary
Near-Doubles	Examples: 6 + 7, 5 + 4
Near-doubles include all additions where one addend is one more than the other. The strategy is to double the smaller number and add one more or double the larger number and take away one. This strategy is useful only for students who know the doubles facts.	
Anchor-of-Ten Facts	Examples: 1 + 9, 2 + 8,
The pairs of whole numbers that add to 10 are foundational facts. Students can easily visualize these pairs using their fingers or beads on a Rekenrek. Students can apply Anchor-of-Ten facts when they decompose larger numbers (e.g., 17 + 23 can be seen as 10 + 20 + (7 + 3) or 10 + 20 + 10).	
Make-Ten Facts	Examples: 3 + 8, 9 + 5
All of these addition facts involve either an 8 or a 9 or both. The idea is to build onto the 8 or 9 to make 10, and then add on the rest. For example, for 8 + 3, start with 8 and add on 2 to make 10, then add on 1 more to get 11. For 4 + 9, start with 9 and add on 1 to make 10, then add on 3 more to get 13.	
In Summary	
With enough practice of strategies and facts, many students will develop automaticity with the above sets of facts, while some students will continue to use the mental strategies for some of these sets of facts. Students who master all of the fact sets named and illustrated above know all of the facts shaded to the right. Notice how few facts have yet to be learned. It should be noted that there are multiple ways to think about adding certain numbers. For example, 8 + 7 can be thought of as a Near-Double (double 7 for 14, plus 1 to get 15) or as a Make-Ten (8 + 2 for 10, plus the remaining 5 to get 15). Teachers should not force the use of any particular strategy.	

Naming and Using Properties

The examples shown for the foundational facts include use of the following properties:

Additive Identity

Zero is called the Additive Identity. Addition or subtraction of 0 leaves the original quantity unchanged, keeping its own identity. Students need opportunities to explore the addition and subtraction of zero. They can think of 3 + 0 as adding nothing on to 3, thus leaving a result of the original 3. They can think of 3 – 0 as taking nothing away from 3, leaving a result of the original 3. Students may say, “Adding 0 doesn’t change the total.”

Commutative Property of Addition

Switching the order in addition yields the same sum. This is called the Commutative Property of Addition. Understanding this property gives students flexibility in solving. For example, think of 3 + 72 as 72 + 3, then find the sum by counting up 3 from 72. Students may say, “The order doesn’t matter when adding.”

Subtraction does not have a commutative property. Students need practice with the idea that you cannot switch the order of numbers in a subtraction question and get the same difference. Using concrete objects, students can reason that “7 take away 4 is 3” makes sense and that “4 take away 7” is quite different.

Associative Property of Addition

This property says that you can group numbers in whatever order you want when adding.

	7 + 6 + 4	Most people find it easier to compute 7 + 10 than
7 + 6 + 4	= 7 + (6 + 4)	13 + 4. The brackets do not have to be included.
= (7 + 6) + 4	= 7 + 10	They are used here to show which numbers are
= 13 + 4	= 17	associated first.
= 17		

The following example shows how the Associative and Commutative Properties can be used together to create more familiar sums.

Example:

1 + 3 + 9 + 23 + 7 = 1 + 9 + 3 + 23 + 7 = (1 + 9) + 3 + (23 + 7) = 10 + 3 + 30 = 40 + 3 = 43

Change the order to make 10s or multiples of 10, and then combine the numbers shown by brackets.

Note that it is not necessary to write the intermediate steps. They are shown here to help illustrate the use of the two properties.

Addition and Subtraction: Connecting Computational Strategies and Properties

Very early, students learn to count by 1s, and learn that the last number named represents the total quantity. The **Counting All** strategy is a natural first approach to addition problems by young students. Example, 3 + 7, the student counts from 0: 1, 2, 3 then 4, 5, 6, 7, 8, 9, 10. A major developmental milestone is moving from counting all to counting on. This typically occurs in Kindergarten or early in Grade 1.

The chart below shows a variety of strategies and its connection to the properties. Many steps are included in the examples for understanding. Not all of these steps would normally be written in solutions. Which of these strategies might be best suited is dependent on the numbers given in the problem.

Strategy	Examples	Additional Information
Count On by 1s	3 + 7 Student starts at first number 3 then counts on 7 numbers by 1s: 3, then 4, 5, 6, 7, 8, 9, 10	Count on one-by-one.
	3 + 7 Student starts at the larger number 7 then counts on 3 numbers by 1s: 7, then 8, 9, 10	Change the order (commutative property) to be able to count on from the larger number.
	10 – 7 Student starts at 7 then counts on by 1s: 7 then 8, 9, 10 The number of numbers counted after 7 to get to 10 is the answer 3.	Count gives the difference.
Count Back by 1s	10 – 7 Student starts at 10 then counts back by 1s 10 then 9, 8, 7 The number of numbers counted back after 10 to get to 7 is the answer 3.	Take away, by 1s until the desired number is reached, while keeping track of how many needs to be taken away.
	10 – 7 Student starts at 10 then counts back by 1s, seven times: 10 then 9, 8, 7, 6, 5, 4, 3 The final number in the count back is the answer.	Take away an amount one-by-one.
Count On or Back by 10s and 1s	79 – 42 Student starts at 79 then counts back by 10s, four times, and then counts back by 1s, two times 79 then 69, 59, 49, 39, 38, 37	Decompose the second number into 10s and 1s and then count back by 10s then 1s.

Strategy	Examples	Additional Information
Add and Subtract the Same Amount (Compensation)	268 + 390 = 268 + (390 + 10) – 10 = 268 + 400 – 10 = 668 – 10 = 658	Add 10, to make a more friendly number, then compensate by taking 10 away.
	861 – 459 = 861 – 461 + 2 = 400 + 2 = 402	Subtract 2 more, to make an easier subtraction, then compensate by adding 2 back on.
	1 000 – 273 = 999 – 273 + 1 = 726 + 1 = 727	Subtract 1 from 1000, to make an easier subtraction, then compensate by adding 1 back on.
Decompose to Make 5s, 10s, or Other Familiar Numbers	8 + 6 = 8 + (2 + 4) = (8 + 2) + 4 = 10 + 4 = 14	Decompose the 6 into 2 and 4, and then use the associative property to add 2 and 8 to make 10.
	3 + 4 = 3 + (2 + 2) = (3 + 2) + 2 = 5 + 2 = 7	Decompose the 4 into 2 and 2, and then use the associative property to add 2 and 3 to make 5.
	597 + 18 = 597 + (3 + 15) = (597 + 3) + 15 = 600 + 15 = 615	Decompose the 18 into 3 and 15, and then use the associative property to add 3 and 597 to make 600.
Decompose by Place Value	53 + 62 = 50 + 3 + 60 + 2 = 50 + 60 + 3 + 2 = 110 + 5 = 115	Decompose numbers by place value and then add the tens and add the ones.
	856 – 325 = 856 – 300 – 20 – 5 = 556 – 20 – 5 = 536 – 5 = 531	Decompose the second number by place value and then subtract by hundreds, tens and ones.
Keep a Constant Difference by Adding or Subtracting the Same Amount to Each Term	655 – 195 = (655 + 5) – (195 + 5) = 660 – 200 = 460	Add 5 to each number to create friendly numbers. The difference will not change.
	102 – 16 = (102 – 2) – (16 – 2) = 100 – 14 = 100 – 10 – 4 = 90 – 4 = 86	Subtract 2 from each number to create a friendly number. The difference will not change.