BIG IDEAS QUESTIONING

K-12

Proportional Reasoning

GAINS

Growing Accessible Interactive Networked Supports in Mathematics Education

BIG IDEAS QUESTIONING K-12 Proportional Reasoning

Big Ideas and Questioning K-12: Proportional Reasoning identifies the key concepts to address the learning in each division. Powerful questions connected to each Big Idea are provided as the means for differentiating instruction while evoking and exposing thinking.

> It is estimated that more than half of the adult population cannot be viewed as proportional thinkers.
> *Susan Lamon, 1999.*

This package is intended for classroom teachers and facilitators of professional learning for mathematics educators.

The materials are posted on the Math GAINS website at www.edugains.ca. It will be updated to include additional sample questions, lessons, and student work.

It is important that the questions we choose help all students to make sense of mathematics, make connections, be successful, and focus on the Big Ideas.

Big Ideas and Questioning, K-12 packages are designed to extend the instructional support provided in TIPS4RM Continuum and Connections. These content-based packages support students' learning by providing:

- questions to develop proficiency connected to the Mathematical Processes,
- rich problems to help develop depth of understanding,
- instructional strategies across Grades 7-10
- planning suggestions.

http://www.edu.gov.on.ca/eng/studentsuccess/lms/files/tips4rm/TIPS4RMccpropreason.pdf

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Proportional Reasoning

Proportional Reasoning involves the deliberate use of multiplicative relationships to compare quantities and to predict the value of one quantity based on the values of another. The term *deliberate* is used to clarify that proportional reasoning is more about the use of number sense than formal, procedural solving of proportions. Students use proportional reasoning in early math learning, for example, when they think of 8 as two fours or four twos rather than thinking of it as one more than seven. They use proportional reasoning later in learning when they think of how a speed of 50 km/h is the same as a speed of 25 km/30 min. Students continue to use use proportional reasoning when they think about slopes of lines and rate of change.

The essence of proportional reasoning is the consideration of number in relative terms, rather than absolute terms. Students are using proportional reasoning when they decide that a group of 3 children growing to 9 children is a more significant change than a group of 100 children growing to 150, since the number tripled in the first case; but only grew by 50%, not even doubling, in the second case.

Although The Ontario Curriculum documents for mathematics do not reference the term *proportional relationships* until Grade 4, activities in the primary grades support the development of proportional thinking. For example, if we ask students to compare the worth of a group of four nickels to the worth of a group of four pennies, we are helping them to develop proportional thinking. In the junior and intermediate grades, students work directly with fractional equivalence, ratio, rate, and percent.

Much formal work on proportion is completed in Grades 9 or 10, but students in higher grades often compare proportional to non-proportional situations. They continue to use proportional thinking when they work with trigonometry and with scale diagrams, as well as in other situations.

Starter Questions for Proportional Reasoning

- Why is Proportional Reasoning important?
- Create both an example and a non-example of proportional reasoning. Try to use unique contexts.
- Describe two different situations where you might want to figure out the unit rate. Tell why knowing the unit rate would be useful in these situations.
- Estimate the number of square centimetres of pizza that all of the students in Toronto eat in one week.
- In parking lot A there are 24 of 40 spots filled. In Lot B there are 56 of 80 spots filled. Which parking lot is fuller?
- Group A: 2 people → 5 people. Group B: 92 people → 100 people. Which group's size changed more?
- Make up your own situation that uses "per" (e.g., Jeremy does 3 good deeds per day). Create a related problem to solve based on your situation.
- The same image is printed in two different sizes, one is 4" × 6" and the other is 5" × 7". Are the pictures exactly alike except for size? If there was a small stick figure in the centre of the smaller picture, how would it be the same or different in the larger picture?
- When I spin a spinner, I am twice as likely to get red as blue and half as likely to get blue as green. What could the probability of green be?

Thinking Absolutely and Relatively: Additive vs Multiplicative Reasoning

- Sam's snake is 120 cm and will be to 270 cm when fully grown. Sally's snake is 150 cm and will grow to 300 cm. Which snake is closer to being fully grown?

  Absolute reasoners: They are both the same distance from being fully grown. Each has 150 cm to grow to be fully grown.

  Relative reasoners: Sally's snake is half way to being fully grown because it is 150 cm and will have to double its length to get to 300 cm. Sam's snake is less than half-way to being fully grown as twice 120 cm is 240 cm and 270 is greater than that (or half 270 cm is 135 cm and it is now only 120 cm – not half grown).

- Two friends mix blue tint with white paint to make blue paint. Nan used more blue tint than Kallam. Nan mixed in more white paint than Kallam. Who mixed the darker shade of blue?

  Additive reasoners: Nan because she put in more blue tint and the more you put in the darker it gets. Or, Kallam because Nan mixed in more white paint than Kallam so that would make Nan's lighter. Or, their paint mixtures would be the same. It says Nan used more blue tint than Kallam but it also says she used more white paint than Kallam so their mixtures seem the same.

  Multiplicative reasoners: You cannot tell whose will be darker because you do not know how much blue tint was added nor how much more white paint was added. All you can tell is that Nan will have more paint. You need to know how many units of blue tint were used for every unit of white paint.

Non-Numeric Problems that Encourage Proportional Thinking

- Carter and Rico like to ride their bikes on the trails in town. Today, they both started riding at the beginning of the trail; each rode continuously at a constant speed, making no stops, to the end of the trail. Rico took longer than Carter to reach the end of the path.

  Which boy was biking faster? How do you know? What are the assumptions? What are the variables?

- The frequency of vibrations of a piano string increases as the length decreases.

  Which piano string would vibrate more slowly, a 90-cm or a 60-cm string? Why? Explain your answer.

- Two carafes of juice are on the table. Carafe B contains weaker juice than carafe A. Add one teaspoon of instant juice crystals to carafe A and one cup of water to carafe B.

  Which carafe will contain the stronger juice? Why?

- Two carafes of juice are sitting on the table. Carafe B and carafe A contain juice that tastes the same. Add one teaspoon of instant juice crystals to both carafe A and carafe B.

  Which carafe will contain the stronger juice? Why?

- Jenson drives 100 km in 2 h and has 60 km to go.

  Will Jensen drive the other 60 km in more or less than 2 h? Justify your answer.

- Greg and Ross hammered a line of nails into different boards from one end to the other. Ross hammered more nails than Greg. Ross' board was shorter than Greg's.

  On which board are the nails closer together? Why?

- If Mary Lou ran fewer laps in more time than she did yesterday, would her running speed be:

  (a) faster? (b) slower? (c) exactly the same? (d) not enough information to tell?

  Explain your thinking.

- Suppose the average height of eighth-grade students is greater than the average height of seventh-grade students.

  Is this an absolute or relative comparison? Why?

Big Ideas

Big ideas are the enduring understandings that underpin the K-12 mathematics curricula. Each one is represented in many, if not all, grades. They are similar to the Mathematical Processes in that they are fundamental content concepts that repeatedly emerge and grow in the study of mathematics.

A focus on Big Ideas assists students in making connections between one thing they learn in mathematics and another. It is the repeated exposure to and discussion of a Big Idea that helps students see its value. Thinking about Big Ideas also assists the teacher in building connections into lesson planning.

Big Ideas from number, measurement, and algebra strands that support proportional thinking:

Reference*	Big Ideas for Number
BIN 1	A number tells how many or how much. *Usually we use numbers to give us the sense of the size of something*.
BIN 2	Classifying numbers or numerical relationships provides information about the characteristics of the numbers or the relationship. *Sometimes if you know a little about a number or relationship, you know more than you realize.*
BIN 3	There are many equivalent representations for a number or numerical relationship. Each representation may emphasize something different about that number or relationship. *There is usually more than one way to show a number or relationship and each of those ways might make something more obvious about that number or relationship.*
BIN 4	Numbers are compared in many ways. Sometimes they are compared to each other; other times, they are compared to benchmark numbers. *Numbers can be compared in different ways – sometimes to each other and sometimes to benchmark numbers.*
BIN 5	The operations of addition, subtraction, multiplication, and division each hold the same fundamental meaning no matter the domain to which they are applied. *The meanings of addition, subtraction, multiplication, and division hold true, regardless of the type of number being used.*
BIN 6	There are many algorithms for performing a given operation. *You can add, subtract, multiply, or divide in more than one way.*

Reference	Other Relevant Big Ideas
BIA 2	Comparing mathematical relationships helps us see that there are classes of relationships with common characteristics and helps us describe each member of the class. *Groups of functions or relationships go together because they behave in similar ways. Knowing about the group helps us know a bit about each member of the group.*
BIM 3	Knowing the measurements of one shape can sometimes provide information about measurements of another shape. *Sometimes two shapes are related so knowing dimensions of one shape allows us to figure out dimensions of the other.*

***BI** – Big Idea; **N** – Number; **A** – Algebra; **M** – Measurement

Based on *Big Ideas from Dr. Small* by Marion Small
(adapted with permission of the author)

BIN 1 A number tells how many or how much.

Usually we use numbers to give us the sense of the size of something.

Some Key Concepts

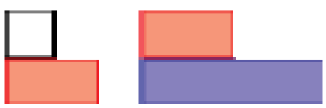
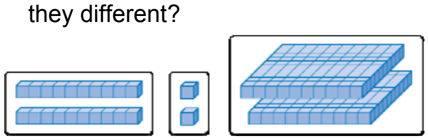
Primary/ Junior	• There are many ways to count. *This could lead to a discussion of skip counting which is a precursor to proportionality.* • A fraction is not meaningful without knowing the size of the whole. $\frac{1}{2}$ *can look really big or really little—it is all about the relationship of the part to the whole.*
Junior/ Intermediate	• The size of a fraction should be thought of as the relationship between its numerator and denominator. *At this level, we want students to realize* $\frac{1}{2}=\frac{2}{4}$ *not because they take the same space in a pie, but because 2 is twice 1 just as 4 is twice 2.*
Intermediate/ Senior	• The size of a radical is often better understood by writing it as a mixed radical. • The sizes of many irrational numbers are better understood by relating them to physical situations (e.g., circles with given radii, triangles with given hypotenuses).

Learning Across Divisions

This is one example of how learning this Big Idea grows across the grades. The lesson goal is grounded in a Curriculum Expectation and connected to the Big Idea. Beginning with the end in mind, a Consolidating Question is developed to determine students' understanding of the lesson goal. (Assessment **for** and **as** Learning).

	Primary	Junior	Intermediate	Senior
Curriculum Expectation	Estimate, count, and represent (using the ¢ symbol) the value of a collection of coins with a maximum value of one dollar. (Grade 2)	Demonstrate an understanding of place value in whole numbers and decimal numbers from 0.01 to 100 000 using a variety of tools and strategies. (Grade 5)	Solve problems involving whole number percents greater than 100. (Grade 8)	Verify, through investigation with and without technology, that $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$, and use this relationship to simplify radicals. (MCR3U)
Lesson Goal	Students are learning to represent amounts of money to 20 cents substituting nickels for pennies and dimes for nickels or pennies.	Students are learning how recognizing the pattern of the place value system helps them figure out how numbers with different numbers of digits are related.	Students are learning to work with percents larger than 100%.	Students are learning to interpret the value of radicals by relating to other radicals.
Consolidating Question(s)	Can 2 coins be worth more than 5 coins?	How do you know that 90 000 has to be 10 times as much as 9000?	Is 150% of a number always larger than 100% of the same number? Justify your answer. 150% of a number is less than 100% of another number. What could those numbers be?	How do you know that $\sqrt{1000}$ must be about 3 × 10?

Questions to Guide and Assess Learning

BIN 1 A number tells how many or how much. *Usually we use numbers to give us the sense of the size of something.*		
Primary/Junior	**Junior/Intermediate**	**Intermediate/Senior**
• Choose two whole numbers less than 100. One should be really big and the other pretty small. Tell how many of the small numbers it would take to make the big one. How do you know? • Choose two decimal numbers less than 1. One should be really big and the other pretty small. Tell how many of the small numbers it would take to make the big one. How do you know? • How are these Cuisenaire rod relationships the same?  • How are these values alike? How are they different?  • What is a good strategy for counting how much the money is worth?  • Twice a number is really big. What do you know about the original number? • There are 4 base ten blocks on the table. What number(s) could they represent? • A number line is labelled starting at 0 and skip counting by 5's. A point on the line is pretty far from one of the numbers and really close to another of the numbers that are labeled. What might it be? How do you know? • Cameron has 7 coins that he says have a value of 17 cents. What could the coins be?	• Choose two fractions less than five. One should be really big and the other pretty small. Tell how many of one fraction it takes to make the other one. How do you know? • How is determining $\frac{2}{3}$ of 12 like determining $\frac{2}{3}$ of 27? How is it different? • A number is just a tiny bit less than halfway between 1 and 2. What could it be? • A store is having a sale. Is it more helpful for you to know that an item is $10 off or 10% off? • How many 2s are in 30? How many 3s are in 45? How can this be? Explain. • You are putting 1 000 bread tags into piles of equal size. Would it take a lot of piles? • Vaughn was comparing numbers. He said he had them in order from least to greatest. This is what he wrote: $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$. Was he correct? Explain.	• Choose two powers (e.g., 2^8). One should be really big and the other pretty small. Tell how many of one power it takes to make the other one. How do you know? • Choose two irrational numbers. One should be really big and the other pretty small. Tell approximately how many of one number it takes to make the other one. How do you know? • Explain why there are an infinite number of fractions between $\frac{8}{10}$ and $\frac{9}{10}$. • You are creating a graph of a function. The minimum value for y is 0 and the maximum for the domain of interest is 422. What scale should you use on the y-axis? Why that scale? • A poll is accurate to within 3 percentage points 19 times out of 20. What do these numbers tell you? • Is a 25% discount the same as a 10% followed by a 15% discount? • If your borrowing interest rate goes up by 1%, does the payment on your loan go up by 1%? Justify your reasoning.

Note: Some questions could address other Big Ideas

BIN 2 Classifying numbers or numerical relationships provides information about the characteristics of the numbers or the relationship.

Sometimes if you know a little about a number or relationship, you know more than you realize.

Some Key Concepts

Primary/ Junior	• To count the number in a group, we often create subgroups and count the number of subgroups. *This leads into a discussion of classifying numbers in terms of the number of digits they have.*
Junior/ Intermediate	• Classifying numbers as factors and/or multiples of other numbers provides additional information about those numbers (e.g., If I classify a number as a multiple of 6, I know lots about it. Multiplication is, of course, the basis for proportional thinking). • Renaming fractions is often the key to comparing them or computing with them. *For example, it is easier to compare $\frac{1}{2}$ and $\frac{5}{8}$ if you think of $\frac{1}{2}$ as $\frac{4}{8}$. You use the proportion $\frac{1}{2}=\frac{4}{8}$ to do the comparison.*
Intermediate/ Senior	• Any radical of the form $\sqrt{m^2 n}$ can be written as an integer multiple of $\sqrt{n}$. • Any number of the form $m\pi$ where m is an integer can be represented as the circumference of a circle with an integer diameter.

Learning Across Divisions

This is one example of how learning this Big Idea grows across the grades. The lesson goal is grounded in a Curriculum Expectation and connected to the Big Idea. Beginning with the end in mind, a Consolidating Question is developed to determine students' understanding of the lesson goal. (Assessment **for** and **as** Learning).

	Primary	Junior	Intermediate	Senior
Curriculum Expectation	Count forward by 1's, 2's, 5's, and 10's to 100, using a variety of tools and strategies. (Grade 1)	Count forward by halves, thirds, fourths, and tenths to beyond one whole, using concrete materials and number lines. (Grade 4)	Generate multiples and factors, using a variety of tools and strategies. (Grade 7)	Evaluate, with and without technology, numeric expressions containing integer exponents and rational bases. (MBF3C)
Lesson Goal	Students are learning to recognize the properties associated with numbers in a skip-counting sequence.	Students are learning to recognize which fractional amounts are actually whole numbers when they count by fractional amounts.	Students are learning to identify factors and multiples of whole numbers.	Students are learning to classify powers by comparing bases and exponents.
Consolidating Question(s)	How do you know that if you skip counted to find the number of socks students in the classroom are wearing today it probably would not be 51?	Two number lines are right under each other. The 0s are lined up. One number line goes up by thirds. The other one goes up by ones. Which numbers on the lines will match up? Why?	Can a number be a factor of one number and a multiple of another number?	Which of these four numbers does not belong? Why? 4^{500}, 2^{1000}, 3^{1500}, and $\sqrt{2}^{2000}$.

Connecting through Questions

BIN 2 Classifying numbers or numerical relationships provides information about the characteristics of the numbers or the relationship.

Sometimes if you know a little about a number or relationship, you know more than you realize.

Primary/Junior	Junior/Intermediate	Intermediate/Senior
• You know that a whole number has a 0 in the ones place. What else do you know about the number? • If you double a number, what do you know about the result? • Fewer than 8 children equally share close to 100 treats. What do you know, for sure, about how many treats each gets? • Eric says that $\frac{8}{8}$ is greater than $\frac{4}{4}$ because there are more pieces. Sylvia says $\frac{4}{4}$ is greater because the pieces are bigger. What do you think? • Ian has some markers. When he puts them into groups of 3, there are 2 left over. If he has fewer than 15 markers, how many markers could he have? • If $\frac{3}{[]}$ is between $\frac{1}{2}$ and 1, what do you know about []? • How does knowing the perimeter of a square help you know other measurements of the square? • If you cut a piece of wood into 3 cm pieces, 1 cm is left over. If it is cut into 4 cm pieces or 5 cm pieces, 1 cm is still left over. How long is the original piece of wood?	• Evan and Leo have the same number of computer games. Evan has organized his into 20 equal piles. Leo has organized his into 5 piles. Do you know anything, for sure, about Leo's piles? Explain. • You know that a fraction is greater than $\frac{2}{3}$ but less than $\frac{5}{6}$. What else do you know about it? • You know that $n \times \Delta = \frac{4}{5}$. What else do you know about n or Δ or other sums, products, quotients, or differences related to the two values? • Does knowing the perimeter of a rectangle help you know other measurements of the rectangle? Explain? • You write a fraction as a decimal and it looks like 0.000 0##. What else do you know about the number? • Imagine an input/output machine. When you input a number that is double another, the output is also doubled by as much. What could the rule be? • π is a number that describes a set of related ratios. What other numbers describe a set of related ratios? • Jiana made an error and multiplied 8 × 36 instead of 9 × 36. How can she find the correct result without starting over? • One decimal is of the form [][][][].[][][] and another is of the form [][][].[][][][]. How are they alike? How are they different?	• A certain angle, θ, in a right triangle has a very big tangent. What do you know about the other trigonometric ratios, sin θ and cos θ? What could the triangle look like and how do you know? • On a spinner, you win 3 points for a red, 4 points for a blue, and 5 points for a yellow, but you lose 3 points for a green and 5 points for a purple. Draw a spinner with five separate regions where the expected value is to lose 1 point. • Why can you use any two points on a line to determine the slope? • Give an example of a linear relationship that is not proportional. • Two quantities have an *inversely* proportional relationship. Explain what this means and give examples. • The fourth term of a geometric sequence is 200. What could the first term be? • How are 4^{500} and 2^{1000} the same and different? Find another pair of numbers that exhibit the same characteristics.

Note: Some questions could address other Big Ideas

BIN 3 There are many equivalent representations for a number or numerical relationship. Each representation may emphasize something different about that number or relationship.

There is usually more than one way to show a number or relationship and each of those ways might make something more obvious about that number or relationship.

Some Key Concepts

Primary/ Junior	• To count the number in a group, we often create subgroups and count the number of subgroups. *This leads into a discussion of alternative place value representations of numbers. It also relates to proportionality in that we are thinking of 2 hundreds as proportional to 2 tens or 2 ones—only the unit changed.* • You can represent a number in a variety of ways. Each representation can focus on a different aspect of the number. *This could lead into a discussion of doubles and maybe doubles strategies.* • There are relationships between all four operations. In particular, multiplication and division are opposites. *This leads into a discussion of the notion that if 8 is four 2s, then 2 is $\frac{1}{4}$ of an 8. When we think of a ratio, we can think of the ratio either way.*
Junior/ Intermediate	• Thinking of numbers as factors or multiples of other numbers provides alternative representations of those numbers. *This could lead into representing products as arrays, and other definitions of and representations of multiplication. Multiplication is the basis for proportional thinking.* • To divide two numbers, you can multiply or divide both by the same amount without changing the quotient. *This leads later to the idea that if* na *is proportional to* nb, *then* a *is proportional to* b. • You can describe the same portion using an infinite number of fractions. *This leads into the notion of equivalence and eventually to a definition of proportion.* • Renaming fractions is often the key to comparing them or computing with them. *It is easier to compare $\frac{1}{2}$ and $\frac{5}{8}$ if you think of $\frac{1}{2}$ as $\frac{4}{8}$. You use the proportion $\frac{1}{2}=\frac{4}{8}$ to do the comparison.*
Intermediate/ Senior	• If you multiply any factor of a number by n, you multiply the product by n. *This leads later to the idea that if* a *is proportional to* b, *then if you double* a, *you double* b. • One can think of a proportion as a statement about alternate ways to represent a ratio. • To divide two numbers, you can multiply or divide both by the same amount without changing the quotient. *If* na *is proportional to* nb, *then* a *is proportional to* b.

Learning Across Divisions

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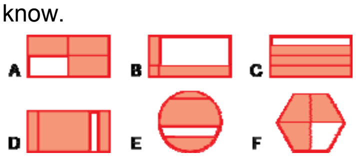
	Primary	Junior	Intermediate	Senior
Curriculum Expectation	Regroup fractional parts into wholes, using concrete materials. (Grade 2)	Determine and explain, through investigation using concrete materials, drawings and calculators, the relationship between fractions (i.e., with denominators 2, 4, 5, 10, 20, 25, 50, and 100) and their equivalent decimal forms. (Grade 5)	Determine, through investigation, the relationships among fractions, decimals, percents, and ratios. (Grade 7)	Determine, without technology, the exact values of the primary trigonometric ratios and the reciprocal trigonometric ratios for the special angles 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, and their multiples less than or equal to 2π. (MHF4U)
Lesson Goal	Students are learning to represent a fraction in more than one way.	Students are learning to represent fractions and decimals in a variety of ways.	Students are learning to recognize which representation – fraction, decimal or percent – is more useful in which situation.	Students are learning to use geometric principles to represent the trigonometric ratios of special angles.
Consolidating Question(s)	Show nine-fourths in a variety of ways. Tell when each way is useful.	A number is less than $\frac{3}{5}$ and greater than $\frac{1}{4}$. Why might it be useful to write the two fractions as decimals?	How would you calculate 25% of 4844? Would you calculate 40% in a similar way? Why or why not?	Represent $\sin\left(\frac{\pi}{6}\right)$ in more than one way. Tell when each way is useful.

Connecting through Questions

BIN 3 **There are many equivalent representations for a number or numerical relationship. Each representation may emphasize something different about that number or relationship.** *There is usually more than one way to show a number or relationship and each of those ways might make something more obvious about that number or relationship.*		
Primary/Junior	**Junior/Intermediate**	**Intermediate/Senior**
• Choose a fraction. Represent it in two different ways. Tell when each representation would be useful. • Represent 20 in many different ways. Which of those ways help you see that 20 is two tens? Which do not? • Jessica said that $\frac{3}{8}$ is less than 1. How did she know that? • How can you arrange your counters to show that 12 is even? How does your arrangement show that? • Which pictures show that $\frac{3}{4}$ of the whole is red? Explain how you know.  • Make 2 rectangle designs with square tiles. Use more than one colour in each design. What fractions do your rectangles show? • Show different ways to cut a sandwich in half. • How much older will you be 1 000 000 seconds from now? (Your answer cannot be in seconds.) • If $\frac{3}{[]}$ is between $\frac{1}{2}$ and 1, what do you know about []? • Horses age more quickly than humans. One horse year is equivalent to about 3 human years. How old would an old horse be? Explain your reasoning. • Mary can walk 67 m in one minute. Can she walk 500 m in 8 minutes?	• Choose a percent greater than 100. Represent it in two different ways. Tell how each representation might be better. • Represent $\frac{4}{5}$ in many different ways. Which of those ways help you see that $\frac{4}{5}$ is the same as 80%? Which do not? • Jane's father drove 417 km in 4.9 hours. Leah's father drove 318 km in 3.8 h. Who was driving faster? By how much? • Describe three situations when it might be useful to know that $\frac{1}{2}$ can be written as an equivalent fraction. • The answer is 10%. What is the question? • What numbers might be missing? $\frac{x}{7} = \frac{a}{30}$ • Two equivalent fractions have denominators that are 10 apart. What can they be? What can they not be? • You want to make a scale drawing of a regular hexagonal patio which is 5 m on a side. What is the largest drawing you can make on a 22 cm × 29 cm piece of paper? • A recipe for 8 muffins uses 2 cups of flour. Choose a different number of muffins – either 10, 12, or 14 muffins. Tell how you will change the amount of flour. How many extra muffins did you make? Why do you not add that many cups of flour to the recipe?	• Choose an angle. Represent its sine value in two different ways. Tell how each representation might be better. • A salary raise results in an increase of about $40 a week on a salary of about $750 a week. Can you predict the increase on a $1000 salary at the same company? Explain. • Andrew said that to calculate $\sqrt{72}$, you should multiply $\sqrt{2}$ by 6. Alyssa said the to calculate $\sqrt{72}$ you should divide 12 by $\sqrt{2}$. Are they both correct? Whose method would you use and why? • The price of an item is greatly reduced but not quite to half. What do you know about the percent discount? • After investing some money at a simple interest rate for 4 years, you end up with $300. How much could you have invested and at what rate? • How does this number line explain why 30% of 85 is about 25?  • Why might it be useful to report fuel efficiency as L/100 km? Would it be just as useful to report it as km/L? • You know that 60% of the students in a school went on a trip. How do you know that the number of students that make up that 60% cannot be exactly 112? • Why would it be useful to write $5^6 \times 20^7$ in a different form to perform the calculation?

Note: Some questions could address other Big Ideas

BIN 4 Numbers are compared in many ways. Sometimes they are compared to each other; other times, they are compared to benchmark numbers.

Numbers can be compared in different ways – sometime to each other and sometimes to benchmark numbers.

Some Key Concepts

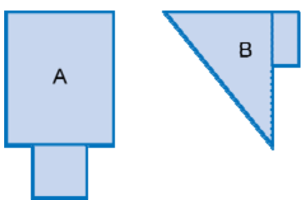
Primary/ Junior	• To compare the numbers of items in two sets, you can match the items, one to one, and see which has more or you can compare the position of the numbers that describe the two amounts in the number sequence. *This could lead into a discussion of comparisons—later we can compare 6 and 3 by thinking of doubling but initially we think of 6 as 3 more.* • Students gain a sense of the size of numbers by comparing them to familiar benchmark numbers. *Later, this leads into the idea that we compare in terms of how many of a number there are, e.g., 27 is about 5 groups of five.*
Junior/ Intermediate	• To compare the numbers of items in two sets, you can use a ratio that focuses on how many of "a" for every "b." *The intention is to focus on multiplicative comparisons.* • You can describe the same portion using an infinite number of fractions. *This leads into the notion of equivalence and eventually to a definition of proportion.* • The size of a fraction should be thought of as the relationship between its numerator and denominator. *At this level, we want students to realize* $\frac{1}{2}=\frac{2}{4}$ *not because they take the same space in a pie, but because 2 is twice 1 just as 4 is twice 2.* • Renaming fractions or ratios is often the key to comparing them or computing with them. *For example, it is easier to compare* $\frac{1}{2}$ *and* $\frac{5}{8}$ *if you think of* $\frac{1}{2}$ *as* $\frac{4}{8}$*. It is easier to compare 2:5 and 3:7 if we write both as percents. This ties to the more general relationships between fractions, decimals, and percents and the notion that we use proportions to find the "friendlier" names.*
Intermediate/ Senior	• To compare two irrational numbers, you might either rename them or compare them using related rationals.

Learning Across Divisions

This is one example of how learning this Big Idea grows across the grades. The lesson goal is grounded in a Curriculum Expectation and connected to the Big Idea. Beginning with the end in mind, a Consolidating Question is developed to determine students' understanding of the lesson goal. (Assessment **for** and **as** Learning).

	Primary	Junior	Intermediate	Senior
Curriculum Expectation	Compare fractions using concrete materials, without using standard fractional notation. (Grade 2)	Represent ratios found In real-life contexts, using concrete materials, drawings, and standard fractional notation. (Grade 6)	Represent the multiplication and division of fractions, using a variety of tools and strategies. (Grade 8)	Determine, through investigation using a variety of tools and strategies the value of a power with a rational exponent. (MCR3U)
Lesson Goal	Students compare fractions using concrete materials.	Students compare ratios in real-life contexts.	Students use the meaning of the operations to compare the sum to the product and the quotient to the difference of two fractions.	Students use the meaning of powers to compare two powers.
Consolidating Question(s)	How could you show that one fifth is less than three fifths?	One group has 3 toddlers and 2 seniors. Another group has 4 toddlers and 3 seniors. Which group would you call "younger"? Why?	How do you know that $\frac{5}{6}\times\frac{2}{3}$ must be less than $\frac{5}{6}+\frac{2}{3}$ without doing any calculations?	How do you know that $4^{\frac{-2}{3}}$ must be greater than $4^{\frac{-4}{3}}$ without doing any calculations?

Connecting through Questions

BIN 4	**Numbers are compared in many ways. Sometimes they are compared to each other; other times, they are compared to benchmark numbers.** *Numbers can be compared in different ways – sometimes to each other and sometimes to benchmark numbers.*	
Primary/Junior	**Junior/Intermediate**	**Intermediate/Senior**
• Choose two fractions with the same denominator. Tell how you know which one is greater. • Compare the numbers 10 and 12. How are they similar? How are they different? • The number of cookies on the plate is closer to two fives than three fives. What might the number be? How do you know? • Jessica said that $\frac{3}{8}$ is less than 1. How did she know that? • One store charges \$4.20 for a package of 4 items. Another store charges \$2.97 for a package of 3 of the same items. Which is a better buy? • How would you compare the numbers 20 and 12? • B is half of A. Agree or disagree?  • If $\frac{3}{[]}$ is between $\frac{1}{2}$ and 1, what do you know about []?	• Choose two fractions with the same numerator. Tell how you know which one is greater. • Jessica said that $\frac{3}{8} < \frac{5}{6}$ since $\frac{3}{8}$ is less than $\frac{1}{2}$ and $\frac{5}{6}$ is more. How did she know that? • One fraction has a numerator and denominator that are 3 apart. Another has a numerator and denominator that are 10 apart. Can you tell which number is greater? Justify. • Jeff's car is going 23 km every 15 minutes. Describe a speed that is just a little bit faster first by changing one of the numbers and then by changing both of them. • Describe two different types of situations where you might want to figure out the unit rate. Tell why knowing the unit rate would be useful. • I am thinking of two fractions really close to 1, but one is a little closer than the other. What might they be? • The following are resting heart rates for animals: Lion – 40 beats in 60 seconds, Giraffe – 5 beats in 12 seconds, Hummingbird – 41 beats in 10 seconds. Which animal has the slowest heart rate? • Compare a pair of given numbers without a calculator: Option 1: 7 and $\sqrt{64}$ Option 2: $\sqrt{196}$ and $\sqrt{220}$	• Choose two angles A and B. Compare the sine of the two angles. Explain. • One map uses a scale of 2 cm represents 43 km. Another uses a scale of 8 mm represents 4 km. How would you estimate to decide which map would make the town look smaller? • Can you ever estimate the square of a number by doubling it? When? • Brand A: 3 cans for \$2.58 Brand B: 2 cans for \$1.59 Why might it be useful to figure out the costs of 3 cans of Brand B? Which one is the better deal? • An irrational number is about 11. What might it be? • How does knowing that $\sqrt{2}$ is approximately 1.4 help you to estimate $\sqrt{32}$? • Order the set of given numbers from least to greatest without a calculator: Option 1: $\sqrt{200}, 2.5^3, 4^2, \sqrt{260}$ Option 2: $\left(\frac{1}{2}\right)^4, 4^{\frac{1}{2}}, 3^{\frac{1}{3}}, 1^8$ • Given the points with co-ordinates $a, b, c, d, e, f, g,$ and h as shown. Which point is closest to ab? $\frac{1}{f}$? $\sqrt{e}$? $\sqrt{h}$? Explain your reasoning.  • Marylou says that $\sqrt{18}$ is twice as large as $\sqrt{9}$. Do you agree or disagree? Why?

Note: Some questions could address other Big Ideas

BIN 5 The operations of addition, subtraction, multiplication, and division each hold the same fundamental meaning no matter the domain to which they are applied.

The meanings of addition, subtraction, multiplication, and division hold true, regardless of the type of number being used.

Some Key Concepts

Primary/ Junior	• There are relationships between all four operations. In particular, multiplication and division are opposites. *This leads into a discussion of the notion that if 8 is four 2s, then 2 is $\frac{1}{4}$ of an 8. When we think of a ratio, we can think of the ratio either way.*
Junior/ Intermediate	• There are relationships between all four operations. In particular, multiplication and division are opposites. *This leads into a discussion of the notion that if 8 is four 2s, then 2 is $\frac{1}{4}$ of an 8. We can think of the ratio either way.*
Intermediate/ Senior	• If you multiply any factor of a number by n, you multiply the product by n. *This leads later to the idea that if* a *is proportional to* b*, then if you double* a*, you double* b.

Learning Across Divisions

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	Primary	Junior	Intermediate	Senior
Curriculum Expectation	Relate multiplication of one-digit numbers and division by one-digit divisors to real-life situations, using a variety of tools and strategies. (Grade 3)	Multiply and divide decimal numbers to tenths by whole numbers using concrete materials, estimation, algorithms, and calculators. (Grade 6)	Represent the multiplication and division of fractions, using a variety of tools and strategies. (Grade 8)	Determine, through investigation with and without technology, some properties of the operations of addition, subtraction, and scalar multiplication of vectors. (MCV4U)
Lesson Goal	Students are learning to use arrays to determine products.	Students are learning that multiplying whole numbers and decimal numbers is fundamentally the same.	Students are learning that multiplying and whole numbers and fractions is fundamentally the same.	Students are learning to use proportional reasoning to justify the magnitude of a vector.
Consolidating Question(s)	How does what you know about multiplication explain why $5 \times 7 = 5 \times 5 + 5 \times 2$?	How does what you know about multiplication help you explain how 4×14 and 4×1.4 are related?	How does what you know about multiplication help you use a picture to show how 7×26 is related to $3\frac{1}{2} \times 5\frac{1}{5}$?	Given vector u is (a,b) and vector v is $(3a, 3b)$. Is the magnitude of vector v three times as large as vector u? Justify your reasoning.

Connecting through Questions

BIN 5 The operations of addition, subtraction, multiplication and division each hold the same fundamental meaning no matter the domain to which they are applied.

The meanings of addition, subtraction, multiplication, and division hold true, regardless of the type of number being used.

Primary/Junior	Junior/Intermediate	Intermediate/Senior
• You double a number. What do you know about the result? • Diane delivers 112 newspapers a week. About how many weeks is it before she delivers 1000 papers? • Ian has some markers. When he put them in groups of 3, there were 2 left over. If he had fewer than 15 markers, how many could he have? • How much older will you be 1 000 000 seconds from now? (Your answer cannot be in seconds.) • Write a note to Mara telling her why you know her answer was not correct before you multiplied. $\begin{array}{r} 234 \\ \times 5 \\ \hline 970 \end{array}$ • Puddings come in packs of 4. Every school day, Sami takes a pudding in his lunch. How does knowing what division means help you figure out how many packs Sami needs for April? • What operation do you use to help you figure out how many different ways you can pack 32 cookies in equal-sized bags? Why that operation?	• How is determining $\frac{2}{3}$ of 12 like determining $\frac{2}{3}$ of 27? How is it different? • Why does it make sense that $2 \div \frac{1}{3} = 2 \times 3$? • How does what you know about subtraction tell you why $\frac{2}{3} - \frac{1}{4}$ is greater than $\frac{1}{3}$ without calculating the answer? • Imagine an input/output machine. When you input a number that is double another, the output is also doubled by as much. What could the rule be? • You know that $n \times \Delta = \frac{4}{5}$. How does what you know about multiplication help you figure out what you know about n or Δ or other sums, products, quotients, or differences related to the two values? • How does what you know about division help you figure out what division question each product checks? a) 13 × 6 + 1 = 79 b) 20 × 6 + 5 = 125 • Why can $\frac{12}{3}$ not have the same value as $\frac{3}{12}$? • Tell how knowing how to multiply helps you to divide. • Tell how 6 × 5 = 30 helps you find $\frac{36}{5}$.	• Why does it make sense that $4 \times \frac{[]}{9} = \frac{[]}{9} \times 4$? • How does what you know about adding help you figure out that $\sqrt{2} + \sqrt{3}$ is not $\sqrt{5}$ but $\sqrt{2} \times \sqrt{3}$ is $\sqrt{6}$? • How is the relationship between these two equations: $\log_2 64 = 6$ and $2^6 = 64$ similar or different to the relationship between 3 × 4 = 12 and 12 ÷ 3 = 4? • Why does it make sense that $\frac{4}{\sqrt{2}} = 2\sqrt{2}$. • Why does it make sense that $2^{\frac{1}{2}}$ is $\sqrt{2}$? • Emma said that 2^{-3} is $\frac{1}{8}$. Jackson said that it is -8. Are they both right? Explain your reasoning. • Can you ever exchange the numerator and denominator of a fraction and get an equal value? If so, when? • When using simple interest, what effect does doubling the rate have on your interest earned? When using compound interest?

Note: Some questions could address other Big Ideas.

BIN 6 There are many algorithms for performing a given operation.

You can add, subtract, multiple, or divide in more than one way.

Some Key Concepts

Primary/ Junior	• Using certain multiplication and division strategies, e.g., half/double, can support development of proportional reasoning. • To divide two numbers, you can multiply or divide both by the same amount without changing the quotient. *For example, this leads later to the idea that if* na *is proportional to* nb, *then* a *is proportional to* b.
Junior/ Intermediate	• Using certain multiplication and division strategies, e.g., half/double, can support development of proportional reasoning. • To divide two numbers, you can multiply or divide both by the same amount without changing the quotient. *For example, this leads later to the idea that if* na *is proportional to* nb, *then* a *is proportional to* b.
Intermediate/ Senior	• Recognizing how multiplication and division relate can lead to more opportunities to use proportional reasoning. • Using certain multiplication and division strategies, e.g., half/double, can support development of proportional reasoning. • To divide two numbers, you can multiply or divide both by the same amount without changing the quotient. *For example, this leads later to the idea that if* na *is proportional to* nb, *then* a *is proportional to* b.

Learning Across Divisions

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	Primary	Junior	Intermediate	Senior
Curriculum Expectation	Multiply to 7 × 7 and divide to 49 ÷ 7, using a variety of mental strategies. (Grade 3)	Divide two-digit whole numbers by one digit whole numbers, using a variety of tools and student-generated algorithms. (Grade 4)	Represent the multiplication and division of fractions, using a variety of tools and strategies. (Grade 8)	Determine the formula for the sum of an arithmetic or geometric series, through investigation, using a variety of tools and strategies and apply the formula to calculate the sum of a given number of consecutive terms. (MCR3U)
Lesson Goal	Students develop strategies to divide by some numbers using knowledge of how to divide by others.	Students develop strategies to divide by some numbers using knowledge of how to divide by others.	Students compare the common denominator and invert and multiply approaches to dividing fractions.	Students use notions of adding numbers in a convenient order to determine the sum of an arithmetic series.
Consolidating Question(s)	Jane said that to divide by 4, you only have to know how to divide by 2. Do you agree? Explain.	Describe three different ways to figure out how many packages of 4 cookies you can make, if you have 84 cookies to package.	Liza said that $\frac{1}{2} \div \frac{1}{6}$ is 3 since there are 3 groups of $\frac{1}{6}$ in $\frac{1}{2}$. Aaron said that the answer is 3 since $\frac{1}{2} \times 6$ is 3. Show that they are really saying the same thing.	Jake decided to calculate the sum of 4 + 7 + 10 + 13 + … + 100 + 103 by adding them to create many subsums that are the same. Why was that a good idea?

Connecting through Questions

BIN 6 **There are many algorithms for performing a given operation.** *You can add, subtract, multiple, or divide in more than one way.*		
Primary/Junior	**Junior/Intermediate**	**Intermediate/Senior**
• Sonia has some cards. Paul has twice as many as Sonia. Trish has twice as many as Paul. If you know how many Sonia has, what different ways can you use to find out how many Trish has? • You want to figure out how many 8s are in 120. What are some different ways you could do that? • You can arrange a batch of almost 400 counters into equal groups. How many groups might there be? • Why is subtracting 6 from 30 five times the same as saying $\frac{30}{6}=5$? • Why is adding 5 seven times the same as sayinging 5 × 7? • If you forget what 8 × 6 is, you can change the factors and start with 2 × 6. Write a series of factors and products you could use to get to 8 × 6 = 48. • How can you answer this question in your head: $\frac{1}{2}$ of 240? • How does knowing how many buns are in one half of a dozen help you figure out how many buns are in one quarter of a dozen? • How is knowing what 4 × 3 is, help you to know what 4 × 6 is? • Tan says to divide by 8, you divide by 4 and then take half. Does this work for any number? Why or why not? • How can you figure out 5 × 17 × 2 in different ways?	• Show at least two ways to simplify $\frac{-4}{5} \div \frac{2}{10}$ • One number is 2.5 times as much as another. What might the numbers be? • Eight muffins cost about $3. Choose a different number of the same kind of muffin and tell how much they cost. • How does knowing 8 × 8 = 64 help you find 8 × 7? • How does 10 × 17 = 170 and 3 × 17 = 51 help you calculate 27 × 17? • You are asked to calculate 25% of 4844. How would you do it? Would you calculate 40% in a similar way or not? • You want to leave a 15% tip for a meal. What are some ways you could calculate the tip? • Which is the better buy for apple juice: 500 ml for $3.50 or 750 ml for $5.00? Justify your answer in more than one way. • Show that $\frac{1}{3}$ of $\frac{1}{4}$ is $\frac{1}{12}$ in more than one way.	• In order to calculate $2^8 \times 5^8$ why is it better to calculate 10^8 rather than by multiplying the two separate powers? • What are some ways to calculate $\log_2 \frac{1}{64}$? • How does $4\sqrt{2}$ compare to $2\sqrt{2}$? Show in more than one way. Which form is preferable? Why? • How can you show in more than one way that $3^{-2}=\frac{1}{9}$? • The product of two radicals is $2\sqrt{3}$. What could the radicals be? Can you find more than pair? • Simplify $\frac{(4\sqrt{6})}{2\sqrt{3}} \times \frac{\sqrt{12}}{(3\sqrt{2})}$ in as many different ways as you can. • How is dividing two irrational numbers, such as $\frac{(2+\sqrt{7})}{(3-\sqrt{2})}$ the same as and different from dividing two complex numbers, such as $\frac{(2+7i)}{(3-2i)}$? • The method of "cross multiplication" is often used to solve a proportion problem. The method is illustrated below. If $\frac{x}{4} = \frac{6}{5}$ then we cross multiply and obtain the equation $5x = 24$ which we solve and obtain $x = \frac{24}{5}$ (or 4.8) Provide a convincing mathematical argument to show that cross multiplication is a valid method for solving a proportion problem.

Note: Some questions could address other Big Ideas

BIA 2 Comparing mathematical relationships helps us see that there are classes of relationships with common characteristics and helps us describe each member of the class.

Groups of functions or relationships go together because they behave in similar ways. Knowing about the group helps us know a bit about each member of the group.

Some Key Concepts

Primary/ Junior	• Comparing growing patterns based on adding the same amount each time can lead to an understanding of how to relate term value to term numbers.
Junior/ Intermediate	• Comparing growing patterns based on adding the same amount each time can lead to an understanding of how to relate term value to term numbers.
Intermediate/ Senior	• An important way to relate two variables is to consider rate of change, i.e., how one variable changes when the other changes. *This leads to the definition of linear relations as ones with a constant rate of change. When we say two variables are proportional, we are talking about linear relations.* • There are two fundamentally different kinds of linear relations – in one situation, the values of one variable are proportional to the corresponding values of the other and, in the other situation, a constant must be subtracted from the dependent variable before its values are proportional to the corresponding values of the independent variable. *This helps distinguish partial and direct variation in terms of whether, if one doubles the x, the y is doubled or not.* • The degree of a polynomial is related to the finite difference where the differences are proportional to the values of the independent variable or proportional to the values of the independent variable if a constant is subtracted. *A quadratic relation is one where the first differences form a linear relation so that the values of those differences are directly proportional to or the values of those differences minus a constant are directly proportional to the values of the independent variable.* • If the general term of an arithmetic sequence is represented as $a + (n - 1)d$, then the values of $(t_n - a)$ are proportional to d. • Variables can be inversely proportional, if the product of their corresponding values is constant. • Certain transformations of graphs can be thought of as creating proportional values, but other transformations cannot.

Learning Across Divisions

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	Primary	Junior	Intermediate	Senior
Curriculum Expectation	Extend repeating, growing and shrinking number patterns. (Grade 3)	Determine the term number of a given term in a growing pattern that is represented by a pattern rule in words, a table of values, or a graph. (Grade 6)	Represent, through investigation with concrete materials, the general term of a linear pattern, using one or more algebraic expressions. (Grade 8)	Determine, through investigation using technology, compound interest for a given investment, using repeated calculations of simple interest, and compare, using a table of values and graphs, simple and compound interest earned for a given principal. (MBF3C)
Lesson Goal	Students are learning how to predict some properties of numbers that will extend a growing pattern by comparing to other patterns.	Students are learning to how to determine the term number for a given value in a linear growing pattern by comparing to a skip-counting pattern.	Students are learning how and why the general term for a linear growing pattern relates to the difference between consecutive values.	Students are learning how simple interest and compound interest formulas differ.
Consolidating Question(s)	How can comparing the pattern 4, 8, 12, 16,… to 5, 9, 13, 17,… help you figure out why all the numbers in the second pattern have to be odd even when you extend it?	How does comparing the patterns 4, 8, 12, 16,… and 5, 9, 13, 17,… help you figure out the 100^{th} number in the second pattern?	How does comparing the patterns 3, 6, 9, 12,... and 2, 5, 8,11,... help you figure out the general term of the second pattern?	Explain why doubling the value of principal invested at 4% simple interest exactly doubles the interest earned, but the same is not true with compound interest.

Connecting through Questions

BIA 2	**Comparing mathematical relationships helps us see that there are classes of relationships with common characteristics and helps us describe each member of the class.** *Groups of functions or relationships go together because they behave in similar ways. Knowing about the group helps us know a bit about each member of the group.*	
Primary/Junior	**Junior/Intermediate**	**Intermediate/Senior**
• How are the kinds of numbers you get when you triple a number different from the kinds you get when you multiply a number by 5? How are they similar? • One bead bracelet is made with 1 red, 3 blue, 1 red, 3 blue, 1 red, 3 blue, and so on. Another is made with 1 red, 2 blue, 1 red, 2 blue, 1 red, 2 blue, and so on. If you use 100 beads, what fraction of each bracelet will be red? Why is red a greater fraction than the other colours? • Choose two numbers less than or equal to twenty. Skip count from 0 to about 100 by each of your chosen numbers. How do the patterns in the ones digits compare? • Here are four number patterns. How are they the same? How are they different? A: 3, 7, 11, 15 B: 4, 8, 12, 16, C: 4, 8, 16, 32 D: 2, 4, 6, 8 • One room has square tables with four chairs each. Another room has triangular tables with three chairs each. What number(s) of students can be accommodated in both rooms with no left overs? Which room uses fewer tables?	• You know that a relationship is created by doubling values and then adding 5 (e.g., 11 becomes 22 + 5 = 27). What other relationship would have virtually the same effect if the original numbers are really big? • You graph $y = 6x$ and $y = 3x$ on different sets of axes. The graphs look identical. How could that have happened? • A pattern is 4, 8, 12, 16,…. A number in this pattern has a ones digit of 2. What could the number be? • Describe some situations that this graph might depict:  • What does a slope of $\frac{8}{5}$ tell you about a straight line? • How are the relations $y = 2x$ and $y = 3x$ alike? How are they different? • Why can you use any two points on a line to determine the slope? • Give an example of a linear relationship that is not proportional. • (2,3), (,), (,), (,), … is a set of ordered pairs in a proportion. What could three additional members of the set be? How could you determine if an ordered pair belongs to the set?	• How are the relations $y = x$ and $y = \frac{1}{x}$ alike? How are they different? • How are the graphs of $f(x) = \frac{1}{x}$ and $f(x) = \frac{4}{x}$ alike and different? • You know that $y = f(x)$ is a linear relationship. If you double the x value, what do you know about the y-value. What else do you know about $f(x)$? • Two quantities have an *inversely* proportional relationship. Explain what this means and give examples. • Where do you think the relations $y = 3x + a$ and $y = 3x + b$ are most alike? • How/why is representing the arithmetic sequence 4, 11, 18, 25,… as $f(n) = 3n + 1$ useful? Is there another way to represent it that is equally useful? • How does $y = 4f(x)$ compare to $y = f(x)$? • Under what conditions will radian measure for an angle be doubled? Justify your answer(s). • A is in quadrant 1 and you know that $\sin(A) > \frac{1}{2}$. What do you know for sure about $3A$ and $\sin 3(A)$? • Which function gets bigger faster: $y = 2^x$ or $y = x^2$. How do you know? • Which variables in the formula $A = P(1 + i)^n$ would you need to set as constant(s) to generate a linear function and an exponential equation. • Which variable(s) in the formula $V = \pi r^2 h$, would you need to set as a constant to generate a linear equation? a quadratic equation? • How is radian measure similar to slope measure? How is it different?

Note: Some questions could address other Big Ideas

BIM 3 Knowing the measurements of one shape can sometimes provide information about measurements of another shape.

Sometimes two shapes are related so knowing dimensions of one shape allows you to figure out dimensions of the other.

Some Key Concepts

Primary/ Junior	• Knowing a length or area of one shape can help you figure out the length or area of a related shape.
Junior/ Intermediate	• Knowing a length or area of one shape can help you figure out the length or area of a related shape. • Using scale diagrams often allows you to use some measurements of a shape to figure out other measurements
Intermediate/ Senior	• Using scale diagrams often allows you to use some measurements of a shape to figure out other measurements. • Knowing two shapes are similar allows you to figure out all the measurements of one shape knowing some of the measurements of the other.

Learning Across Divisions

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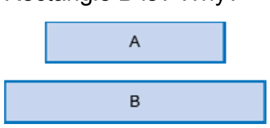
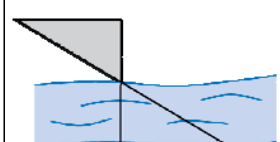
	Primary	Junior	Intermediate	Senior
Curriculum Expectation	Compare two or three objects using measurable attributes and describe the objects using relative terms. (Grade 1)	Determine, through investigation using a variety of tools and strategies, the relationship between the area of a rectangle and the areas of parallelograms and triangles, by decomposing and composing. (Grade 6)	Solve problems involving ratios, rates, and directly proportional relationships in various contexts using a variety of methods. (MFM1P)	Solve problems, including those that arise from real-world applications by determining the measures of the sides and angles of right triangles using the primary trigonometric ratios. (MCF3M)
Lesson Goal	Students are learning how to determine the measure of a shape using the measure of another shape.	Students are learning how to determine the measure of a shape using the measure of another shape.	Students are learning how to determine the measure of a shape using the measure of another shape.	Students are learning how to determine the measure of a shape using the measure of another shape.
Consolidating Questions	You measured the long side of Rectangle A and it was 8 rods. How long do you think the long side of Rectangle B is? Why? 	A rectangle and a parallelogram share a base. The parallelogram is twice as tall. If you know the area of the rectangle, can you figure out the area of the parallelogram?	On a scale drawing two towns that are 50 km apart are only 3 cm apart. If you know how far apart two towns are on the drawing, can you be sure how far apart they really are? Explain.	Which measurements of the shaded triangle do you need to know to help you figure out the width of the river? Why those measurements? 

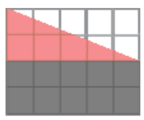
Connecting through Questions

BIM 3 Knowing the measurements of one shape can sometimes provide information about measurements of another shape.

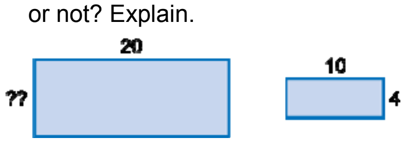
Sometimes two shapes are related so knowing dimensions of one shape allows you to figure out dimensions of the other.

Primary/Junior

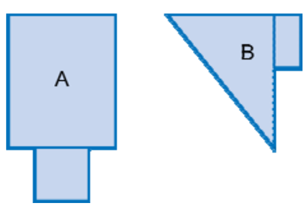
- Consider the shapes on the grid. How does knowing the area of the darker rectangle is 40 cm^2, help you determine the area of the red triangle?



- It takes 12 long rods to measure the width of a table. How many short rods will it take?
- Can you predict the missing width or not? Explain.

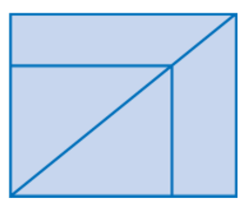


- B is half of A. Agree or disagree?



Junior/Intermediate

- Ian created a scale drawing of a house. The house is really 7 m high; it is only 30 cm high on the scale drawing. If the house is really 13 m wide, can you be sure how wide it would be on the scale drawing? Explain.
- Ross has a picture on his computer of a rectangle that is twice as long as wide. If he enlarges the shape so that the length is 50 cm, what is the width?
- The length of one rectangle is double the length of another. Their widths are the same. What happens to other measures of the rectangle – are they doubled too? Explain.
- One circle has double the radius of another. What is true about the relationship of their perimeters? areas?
- Why is it always true that if two rectangles are similar their diagonals will match up when they are stacked one on top of the other.*****



- How can we find the key measurements needed to properly design a shirt for a doll if we knew the circumference of the doll's wrist?
- If you found a student's shoe, how might you determine that student's height?
- How can you use graph paper to enlarge a picture?
- How can you determine the height of an object from its shadow?

Intermediate/Senior

- Jeff said that he has created an accurate scale diagram of your rectangular classroom. How many and which measurements do you have to take to decide if he is correct?
- One right triangle is similar to another. Two of the sides of the smaller triangle are 3 and 8. One of the sides of the larger triangle is 6. What could all the side lengths of all three triangles be?
- The tangent of an angle in right triangle A is triple the tangent of an angle in right triangle B. Is one angle triple the size of the other?
- How does doubling the diagonal of a square affect its perimeter and area?
- Sarah claims that when two triangles have one angle the same size, then the triangles have proportional sides. Do you agree? Why or why not.
- One cylinder has double the volume of another. How could the radii and heights be related?
- How is radian measure similar to similar triangles? How is it different?

* Sample solutions are available in the Continuums and Connections Proportional Reasoning Package
http://www.edu.gov.on.ca/eng/studentsuccess/lms/files/tips4rm/TIPS4RMccpropreason.pdf

Note: Some questions could address other Big Ideas

Differentiating Instruction Through Questioning

One of the most effective means to address the variety of student needs is through the questions teachers pose in instruction and assessment. Good questions should focus on important mathematics, engage every learner, and offer possibilities for each learner to participate in a rich mathematical conversation, no matter what their stage of learning. The questions teachers ask further the mathematical development of students by exposing student thinking and, through the ensuing dialogue, provide teachers with information needed to adjust their intended instructional trajectory. For good questions to work, students must be provided with the appropriate scaffolding and challenge in an environment that allows them to speak and listen fearlessly. Good questions provide opportunities for rich mathematical conversations thus promoting math talk learning communities.

By providing choice, teachers allow students to work at their readiness level while working on a common learning goal. Open questions and parallel tasks are two core questioning strategies that allow for choice.

Open Questions

Open questions allow for a variety of responses or approaches. Students choose the numbers or mathematical models they are comfortable with, thus promoting confidence.

For example:
Question A: A pair of shoes regularly sells for $60. They are on sale for 25% off. What is the sale price?
Question B: A pair of shoes is on sale for 25% off. What could the original and sale price be?

Question B is open because students can choose the numbers with which they would like to work. This choice makes it more inclusive and accessible to a larger range of student abilities. This choice also allows students to demonstrate what they do know about working with percents and not be hampered by values with which they are not comfortable.

Open questions can be used in the beginning of a lesson, **Minds On…**, to engage students in new learning, build on students prior knowledge, or serve as assessment **for** learning opportunities. They could be used in the middle part of the lesson, **Action!,** for students to explore new ideas at their readiness level, or in the third part of the lesson, **Consolidate**, focusing on assessing student understanding of the goal for the lesson.

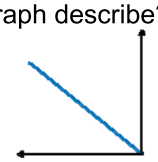
Strategies for Opening a Question

- Begin with the answer. Ask for the question.
  Example: You saved $6 on a pair of jeans. What could the original price and the percent off have been?
- Ask for similarities and differences.
  Example: How are $y = 3x$ and $y = 2x$ alike? How are they different?
- Leave certain information out of the problem, e.g., omit numbers.
  Example: Two right triangles are similar. One has two side lengths of 4 and 6. The other has one side length of 12. What lengths could the other three sides be?
- Provide several numbers and math words; the student creates a sentence using all the numbers and words.
  Example: Create a sentence that uses the numbers and words: 40, 5, ratio, and scale.
- Use “soft” language.
  Example: Two ratios are “almost but not quite” equivalent. What might they be?

Scaffolding Questions

Meaningful learning occurs during problem solving, when students do not know the answer or what to do and have to investigate for some period of time. All students should be encouraged to explore for a bit, but some students do not even know how to start. Teachers need to be prepared to scaffold the learning for these students with question prompts, when they are clearly needed. The following scaffolding questions are designed to preserve the cognitive demand of the original question rather than break the original question into a multi-step procedure.

Examples

Open Question	Scaffolding Questions
One number is 2.5 times as much as another. What might the numbers be?	• Which number is bigger? How do you know? • If the first number is less than 10, what do you know about the second number? • Does either number have to be a decimal?
An item costs more than $60. You saved $35. What might the percent discount have been?	• Suppose you had saved 50%. What do you know about the original price? • Suppose you had saved 25%. What do you know about the original price? • For what price would be easy for you to figure out the percent discount?
What situations could this graph describe? 	• Suppose you use two variables. If one increases, what should the other do? How do you know? If one doubles, what should the other do? How do you know? • How would it help to put numbers on the axes?

Parallel Questions

Another strategy that builds student confidence is to offer choice in two or three parallel tasks that are mathematically equivalent in terms of the learning concept but not in terms of skill details. All students can succeed relatively independently since obstacles are removed in the varied tasks to provide an entry point for all students. The context of the tasks are relatively similar so that the same follow-up questions can be asked and answered by students no matter which task they did. This differentiation technique allows all students to be part of the discussions and encourages a math talk learning community.

Steps for Creating Parallel Questions

1. Select the initial task.
2. Anticipate student difficulties with the task (or anticipate what makes the task too simple for some students).
3. Create the parallel task, ensuring that the Big Idea is not compromised, and that enough context remains similar so that common consolidation questions can be asked.
4. Create at least three or four common questions that are pertinent to both tasks. These should provide insight into the solution and not just extend the original tasks. You might use processes and Big Ideas to help.
5. Call upon students from both groups to respond.

Sample Parallel Tasks

Choice 1	Choice 2	Common Questions	Scaffolding Questions
Ian has some markers. When he puts them in groups of 3, there are 2 left over. If he has fewer than 15 markers, how many markers could he have?	Andrea has some markers. When she put them in groups of 3, there is 1 left over. When she put them in groups of 4, there are 3 left over. If she has fewer than 20 markers, how many markers could she have?	• Could your number be 3? 4? • How did you decide what numbers to try? • How did you solve the problem? • Would there be more answers, if we do not limited the number of markers? • How would you find those additional answers?	• Here are some markers. How would you show that there is 1 (or 2) left over when you make groups of 3? • How do you know there are more than 4 markers? • Are there other numbers you could eliminate?
Two fractions are equivalent. If you add the numerators, the result is 22 less than if you add the denominators. Draw pictures of the two fractions.	Draw a picture to show two equivalent fractions for 2/8.	• Could one of your fractions be represented as $\frac{1}{4}$? • Could one of your fractions be $\frac{1}{2}$? • Do your pictures prove your fractions are equivalent? How would you change them so they do? • What were your fractions?	• How do you know the fractions are less than 1? • What makes fractions equivalent? • Suppose you have a picture of one fraction. How can you change the picture to get an equivalent fraction?
About how many years old is someone who is 1000 days old?	About how many years old is someone who is 1 000 000 seconds old?	• Could you have predicted whether or not the person was more than a year old? Explain. • What facts did you need to use to help you solve the problem? • Did you estimate or calculate? How? • About how far off do you think your estimate is? Why?	• How many days are in a year? • To estimate, would you change the 1000 or the 365? Why? • Would it help or not to figure out how many hours are in a week? • Would it help or not to figure out how many seconds are in a month?

Sample Parallel Tasks (continued)

Choice 1	Choice 2	Common Questions	Scaffolding Questions
A number between 20 and 30 is 80% of another number. What could the second number be?	A number between 20 and 30 is 150% of another number. What could the second number be?	• Is the second number greater or less than the first one? How did you decide? • Is there more than one answer? How do you know? How far apart are they? • What strategy did you use? • How else could you compare the two numbers?	• How else can you think of 80%? 150%? • How do you know that the second number cannot be 50? • What picture could you draw to help you? • What is the least the second number could be? How do you know?
500 mL of Brand A juice costs $2.29, but 620 mL of Brand B juice costs $2.69. Which is a better buy?	500 mL of Brand A juice costs $2.49, but 750 mL of Brand B juice costs $3.29. Which is a better buy?	• Why does it make sense that the price for Brand B is not a whole lot higher than it is? • Did you need to calculate exactly or could you estimate to decide which was a better buy? • Did you figure out the unit price to decide which was a better buy? Did you need to? • How did you solve the problem?	• How does 620 (or 750) compare to 500? How is that information useful? • About how much does 100 mL of Brand A cost? How is that information useful to you? • How could a diagram help you solve the problem?
On a scale diagram, 1 cm represents 10 km. If two places are 3.5 cm apart on the map, how far apart are the actual places?	On a scale diagram, 1 cm represents 7.5 km. If two places are 3.5 cm apart on the map, how far apart are the actual places?	• How far apart are the towns? • How did you figure it out? • Why did the scale only have to tell about 1 cm and not 2 cm or 3 cm? • Which was the easiest part of your calculation? • Which part was the most difficult?	• How much would 2 cm represent? 3 cm? .5 cm? • Would a diagram be helpful? • Will your answer be larger than 4? Approximately how many times larger?
Two variables are almost proportional. What might the relationship be? How do you know?	Two variables are almost inversely proportional. What might the relationship be? How do you know?	• What relationship did you think about first? Why? • Did you use that first relationship? Why not? • How did you interpret “almost”? • What did you notice about your graph?	• What does *variable* mean? • What sort of answer do you need? • What does proportional mean? • If x gets bigger, what happens to y? Why? • How could a graph help?

Lesson Planning: Questions in the 3-part Lesson

Minds On… Questions	Action! … Questions	Consolidation … Questions
• are engaging to get students hooked • serve as assessment **for** learning opportunities • could be open questions, building on what students already know	• are more substantive and activate a problem/task/ exploration that requires students to confront the new knowledge that is the goal of the lesson • may require scaffolding questions that are more specific but still as open as possible • could be open questions or parallel tasks	• are planned first, i.e., beginning with the end in mind • focus on connecting to the Big Idea for the lesson • could be open questions, focusing on assessing student understanding of the goal for the lesson
Examples: I was to find the value of a group of base ten blocks. When I counted them, I said 4 numbers. What numbers might I have said? I am thinking of two fractions really close to 1, but one is a little closer than the other. What might they be? A proportion is solved by substituting $x = 22$. What might the proportion be? If an angle is three times larger, what happens to its radian measure?	Examples: Imagine an input/output machine. When you input a number that is double another, the output is also doubled. What could the rule be? Choice 1: Two equivalent fractions have denominators that are 10 apart. What could they be? What cannot they be? Choice 2: Two equivalent fractions have denominators that are 3 apart. What could they be? What can they not be? Choose one of the solids below. What happens to the volume of the solid if you double one or two of its dimensions?  The corresponding hypotenuse lengths of two similar right triangles are exactly 1 cm apart in length. What could all the side lengths be?	Examples: A number line is labeled starting at 0 skip count by 5s to label the line. A spot on the line is pretty far from 0 and really close to one of the numbers that is labeled. What might it be? How do you know? Why is every second multiple of 6 also a multiple of 4? Jeff's car is going 23 km every 15 minutes. Describe a speed that is just a little bit faster first by changing one of the numbers and then by changing both of them. How are angle measure and radian measure alike and different?

Lesson Planning: Posing Powerful Questions

This Posing Powerful Questions Template is an adaptation of the TIPS Three-Part Lesson Template with a focus on rich questions for each part of the lesson. Beginning with the end in mind, the lesson goal is filtered through a curriculum expectation connected to the Big Idea and a Consolidating Question is developed to determine students' understanding of the lesson goal. (Assessment **for** and **as** Learning)

Note: A blank planning template is available on the Math GAINS website at www.edugains.ca.

Lesson Title **Grade/Program:**

Goals(s) for a Specific Lesson
Curriculum Expectations
Big Idea(s) Addressed by the Expectations
Minds On… **Sample Question(s):**
Action! **Sample Question(s):** **Scaffolding Questions:** *(posed to individuals as needed)*
Consolidate/Debrief **Sample Question(s):**

Lesson Title: Counting **Grade: 1**

Goals(s) for a Specific Lesson
Students are learning to use a variety of strategies to count objects up to 100 and explain their strategy choices.

Curriculum Expectations
Count forward by 1's, 2's, 5's, and 10's to 100, using a variety of tools and strategies (e.g., move with steps; skip count on a number line; place counters on a hundreds chart; connect cubes to show equal groups; count groups of pennies, nickels, or dimes).

Big Idea(s) Addressed by the Expectations
A number tells how many or how much. (BIN 1)
Usually we use numbers to give us the sense of the size of something.

Minds On…
Sample Question(s)
Give pairs of students small bags (30) of blocks (or other objects) and ask how many blocks are in the bag.
Have them count and then ask students how they counted.
Coordinate discussion prompting for different ways of counting by 1's, 2's, 5's, and 10's.
Practise counting by 2's, 5's and 10's, using the hundreds chart.

Action!
Sample Question(s)
Option 1: Jelly Bean Counting Contest. Your guess is 63. How would you count the jellybeans to see how close your guess is to the number of jelly beans in the jar?
Option 2: Your school is collecting pennies for donations to the food bank. Mom says there are 25 pennies in a jar that you may have. How will you count them to make sure she is right?

Scaffolding Questions
(posed to individuals as needed)
How can you arrange the cubes to make them easier to count?
Show me what you have done so far.

Consolidate/Debrief
Sample Question(s)
How would you organize about 100 people to make counting the exact number of people in the room more manageable?
List the numbers you would say to get to your total count.
Why did you choose to count that way?
How is your way similar to your friend's way of counting?
What makes it easier to count lots of things?

Lesson Title: What if the Numbers Change? **Grade: 5**

Goals(s) for a Specific Lesson
Students are learning how to justify a decision as to whether to compare numbers additively or multiplicatively.

Curriculum Expectations
Describe multiplicative relationships between quantities by using simple fractions and decimals (e.g., If you have 4 plums and I have 6 plums, I can say that I have $1\frac{1}{2}$ or 1.5 times as many plums as you have.)

Big Idea(s) Addressed by the Expectations
Numbers can be compared in different ways. Sometimes they are compared to each other and sometimes to benchmark numbers. (BIN 4)

Minds On…
Sample Question(s)
Aaron has 24 apples in his basket. This is 4 times as many apples as Akeem. How many apples does Akeem have?
Show a variety of representations, grouped to make 4 groups.

Action!
Sample Question(s)
The food services manager wants to order $1\frac{1}{2}$ times the number of apples for every student who eats at the cafeteria. If the cafeteria services 400 students, how many apples should the manger buy? How do you know?

Scaffolding Questions
(posed to individuals as needed)
How could you represent the number of apples for every 4 students? For every 40 students? For every 400 students?
What if they wanted to buy 1 apple for each student? Or 2 apples for each student? Or half way between: $1\frac{1}{2}$ apples for each student?

Consolidate/Debrief
Sample Question(s)

a) A recipe for 8 muffins uses 2 cups of flour. Choose a different number of muffins – either 10, 12, or 14 muffins. Tell how you will change amount of flour.

b) How many extra muffins did you make? Why do you not add that many cups of flour to the recipe?

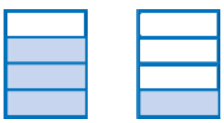
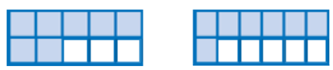
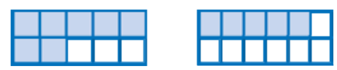
Lesson Title: How Can I Compare Fractions? **Grade: 6**

Goals(s) for a Specific Lesson
Students are learning to compare fractions by either comparing them directly or using benchmarks.

Curriculum Expectations
Represent, compare, and order fractional amounts with unlike denominators, including proper and improper fractions and mixed numbers, using a variety of tools (e.g., fraction circles, Cuisenaire rods, drawings, number lines, calculators) and using standard fractional notation.

Big Idea(s) Addressed by the Expectations
Numbers can be compared in different ways. Sometimes they are compared to each other and sometimes to benchmark numbers. (BIN 4)

Minds On…
Sample Question(s)

1) Which is greater $\frac{3}{4}$ or $\frac{1}{4}$? 	2) Which is greater $\frac{7}{10}$ or $\frac{7}{12}$? 	3) Which is greater $\frac{7}{10}$ or $\frac{5}{12}$? 

Action!
Sample Question(s)
Which chore took longer to complete?

a) $\frac{4}{5}$h doing laundry or $\frac{2}{5}$h vacuuming?

b) $\frac{3}{10}$h washing dishes or $\frac{3}{12}$h drying dishes?

c) $\frac{1}{2}$h collecting garbage or $\frac{3}{5}$h cleaning the bathroom?

Scaffolding Questions
(posed to individuals as needed)
When the denominators are the same, how can you tell which fraction has the lesser value?
When the numerators are the same, how can you tell which fraction represents the lesser value?
When both numerators and denominators are different how can you compare the fractions to benchmarks, such as 0, $\frac{1}{2}$ and 1?

Consolidate/Debrief
Sample Question(s)
Question 1: Agree or disagree. The best way to compare the fractions $\frac{3}{5}$ and $\frac{5}{9}$ is to draw a picture. Explain your thinking.

Question 2: Which is greater: $\frac{3}{5}$ or $\frac{5}{9}$? How do you know?

Lesson Title: Ratio Tables Show Equivalent Ratios **Grade: 9 Applied**

Goals(s) for a Specific Lesson
Students are learning to recognize that a ratio table to generate equivalent ratios is a strategy to solve a proportion.

Curriculum Expectations
Solve for the unknown value in a proportion, using a variety of methods (e.g., concrete materials, algebraic reasoning, equivalent ratios, constant of proportionality).

Big Idea(s) Addressed by the Expectations
There is usually more than one way to show a number or relationship and each of those ways might make something more obvious about that number or relationship. (BIN 3)

Minds On…

Sample Question(s)
Write three equivalent ratios. How do you know they are equivalent?

Debrief: Introduce ratio tables. i.e.,

1	2	5
2	4	10

See the Glossary.

Action!

Sample Question(s)
Fill in the missing values in the ratio tables:

2	4		10	5	1
3		9			

3				1	5
4	2	1	5		

8		4		10		
5	10					7

Scaffolding Questions
(posed to individuals as needed)
What number can you multiply or divide by to get equivalent fractions?
Which ratio would you use to help you get the missing value? (Students select any pair of ratios in the table with which to work.)
Would it be helpful if one of your ratio terms were 1? Which term? Which ratio would you use to make that happen?
Note: A ratio entry of 1 is useful for calculations where the multiplication or division is not obvious.

Consolidate/Debrief

Sample Question(s)

3:x = 4:10
How can you solve for x? What would x be?

Lesson Title: Comparing Simple Interest **Grade: MEL3E**

Goals(s) for a Specific Lesson
Students are learning how simple interest amounts are affected when one of the variables in a simple interest calculation is changed.

Curriculum Expectations
Determine, through investigation using technology, the effect on simple interest of changes in the principal, interest rate, or time, and solve problems involving applications of simple interest

Big Idea(s) Addressed by the Expectations
Comparing mathematical relationships helps us see that there are classes of relationships with common characteristics and helps us describe each member of the class. (BIA 2)

Minds On…
Sample Question(s)
Ian was calculating the amount of interest he earned in his bank account which pays simple interest. He wrote: I = 100(0.02)(3). What do each of his numbers represent?

Donna earned more interest. Her calculations look almost the same. What could her interest calculation look like? How much did she earn?

Action!
Sample Question(s)
Scenario 1: You are thinking about investing an amount of money for several years at a simple interest rate less than 5% per year. How much interest will you earn?

Examine how the amount of interest you earn is affected if you:

Scenario 2: double one of the variables – principal, rate, or time?
Scenario 3: double any two of the variables – principal, rate, or time?
Scenario 4: double all of the variables – principal, rate, and time?

Scaffolding Questions
What formula should you use?
What values will you use for the principal, rate, and time?
What would be a useful representation of your interest rate for calculations?
Would a table be useful? Why or why not?
Describe in words the relationship between the interest earned in Scenario 1 and the interest earned in each of the other scenarios.

Consolidate/Debrief
Sample Question(s)
Choose an amount of money to invest at a fixed simple interest rate for a period of time.
Show, in as many ways as you can, how you could change the other variables to earn six times the amount of interest.

Lesson Title: Comparing Functions **Grades: MCR3U, MCF3M, MBF3C**

Goals(s) for a Specific Lesson
Students are learning to identify a relationship as linear, quadratic, or exponential by recognizing that a linear function grows at a constant rate (constant first differences), that quadratic functions grow at an increasing rate (constant second differences), and that exponential functions have a common ratio between consecutive y values.

Curriculum Expectations
MCR3U: distinguish exponential functions from linear and quadratic functions by making comparisons in a variety of ways, within the same context when possible.
MCF3M, MBF3C: distinguish exponential relations from linear and quadratic relations by making comparisons in a variety of ways, within the same context when possible.

Big Idea(s) Addressed by the Expectations
Comparing mathematical relationships helps us see that there are classes of relationships with common characteristics and helps us describe each member of the class. (BIA 2)

Minds On…
Sample Question(s)
Create two tables of values; one that shows a constant rate of change and one that shows an increasing rate of change. How do your tables show each of these rates?

Action!
Sample Question(s)
Explain in a variety of ways how you can distinguish the exponential function $f(x) = 2^x$ from the quadratic function $f(x) = x^2$ and the linear function $f(x) = 2x$.

Scaffolding Questions
(posed to individuals as needed)
What are some of the ways you can compare the relationships?
At what different representations can you look for each of the relationships?
How are the table of values for the relationships similar? How are they different?
How are the graphs similar? How are they different?
How are the equations similar? How are they different?

Consolidate/Debrief
Sample Question(s)

1. Classify each of the following relations as linear, quadratic, or exponential.

   a) $y = 3x + 1$ b) $y = 3x^2 + 5$ c) $y = -0.05x$ d) $y = 1.2^x$

2. Create a flowchart that you (or a classmate) could use to determine whether a relationship was linear, quadratic, or exponential.
3. Predict whether a quadratic, exponential, or linear model would best fit each situation.
   a) the position of a person running straight at a constant speed is 2.5 m/s
   b) the population of a bacteria culture doubling every 30 minutes
   c) the height of a football over time after being thrown to another player
   d) the population of a city that has been increasing at a rate of 3% each year
4. Rex Boggs had a hypothesis that the daily weight of the bar of soap was not a linear function because the little bit at the end seemed to last forever. So, Rex recorded the weight of his soap most days during a month. Examine the data and determine whether it is linear, quadratic, or exponential and justify your reasoning.

Day	0	1	4	5	6	7	8	9	11	12	17	19	20	21	22
Mass (g)	124	121	103	96	90	84	78	71	58	50	27	16	12	8	6

Instruction Connections

Suggested Instructional Strategies	Helping To Develop Understanding
Grade 7 • Investigate and represent the relationships among fractions, decimals, and percents, using concrete materials. • Connect students' knowledge about percents to make comparisons in context. • Express probabilities as ratios. • Apply both experimental and theoretical probabilities to make predictions. • Investigate dilatations, using a variety of tools, e.g., pattern blocks, overhead transparencies, computer technology. • Create similar triangles on the Cartesian plane using grid paper, measure and compare angles, longest, shortest, and remaining sides. • Enlarge/reduce similar triangles to build understanding of the proportionality of similar triangles, using The Geometer's Sketchpad®4. • Model rates by graphing to visualize the relationship, and by creating algebraic expressions. • Illustrate ratios, using scale drawings. • Model situations requiring the determination of best rates, using geoboards. **Grade 8** • Investigate the relationship between the circumference and the diameter of a circle as a ratio to determine π. • Solve percent problems arising from everyday familiar contexts in more than one way. • Solve problems involving proportions, using concrete materials and drawings. • Develop an understanding that proportions are multiplicative relationships. • Represent probabilities in multiple ways, e.g., fractions, percents, ratios. • Investigate similar figures using concrete materials and The Geometer's Sketchpad®4. • Investigate proportional relationships among measurable attributes of geometric figures. **Grade 9 Applied** • Explore and develop an understanding of proportions, using a variety of contexts, tools, and methods. • Investigate ratios using examples of proportional and non-proportional situations to arrive at an understanding of the multiplicative relationship of proportions. • Investigate a variety of methods for solving problems involving proportions, e.g., ratio tables, drawings, graphs, unit rates, scaling factors. • Support students in developing proportional reasoning, and procedural fluency in its application. **Grade 10 Applied** • Investigate the properties of similar figures, using concrete materials, e.g., geoboards, The Geometer's Sketchpad®4. • Convert imperial measurement and metric measurement, using proportions. • Solve problems involving similar triangles in familiar contexts, using a variety of methods.	• When examining two ratios, it is sometimes useful to think of them as being either *within* ratios or *between* ratios. A ratio of two measures in the same setting is a *within* ratio. A *between* ratio is a ratio of two corresponding measures in different situations. (John Van de Walle) Provide opportunities for students to think about ratios in these two ways. • A ratio can compare a part to a whole (part-to-whole) or a part of the whole to another part of the whole (part-to-part). Provide opportunities for students to develop their understanding by setting activities in a wide range of contexts. • Demonstrate that ratios can also compare measures of two different types. (rates) • Relate proportional reasoning to existing processes, for instance the concept of unit fractions is similar to unit rates. • Delay the use of rules and algorithms for solving proportions, until students have developed proportional reasoning skills. • Encourage discussion and experimentation in comparing ratios. • Vary the numerical relationships or the context in which the problems are posed. • Pose problems for students to explore that help them to identify multiplicative situations. • Fraction, ratio, and rational number ideas are mathematically complex and interconnected. Provide students with time to construct important ideas and ways of thinking. (Lamon) • Give students multiple opportunities to progress through different representations – concrete → diagrams/tables → symbolic. • Have students write word statements to describe their reasoning. • Have students represent proportion problems as unit rates. • Use numerical comparison problems where two complete rates are given and students are asked to answer questions about how the rates are to be compared. • Present problems that involve qualitative prediction and comparison that require responses not dependent on specific numerical values. • Research suggests teaching multiple strategies, starting with unit rate and factor of change which are more intuitive then followed by fractions and the cross product algorithm.

Appendix 1

Self-Reflection: Frayer Model

A Frayer model is a visual organizer for key words and concepts.
Students can benefit from creating Frayer models using their own pictures, numbers, and words.
For more information on Frayer models download:
Think Literacy: Subject-Specific Examples Mathematics Grades 7-9
http://www.edu.gov.on.ca/eng/studentsuccess/thinkliteracy/library.html#subjects

Proportional Reasoning

Definition

Proportional reasoning involves the deliberate use of multiplicative relationships to compare quantities and to predict the values of one quantity based on the values of another.

It is assumed that proportional reasoning involves conceptual and not just procedural thinking.

Facts/Characteristics

- thinking about quantities and change
- comparing using multiplication/division
- using the same “whole” for comparing fractions
- using rational numbers
- thinking about multiplicative relationships
- partitioning
- unitizing
- making sense of ratios
- using relative thinking (as opposed to absolute thinking)

Examples

Thinking about …

- skip counting
- side lengths in similar triangles
- scale factors
- rates, unit rates, interest rates
- solving proportions
- linear relationships that have direct variation

Reasoning that …

- if a car is traveling at a rate of 90 km/h, it will take 2.5 h to travel 225 km
- $\frac{18}{37}$ is a bit less than $\frac{1}{2}$ because 18 is a bit less than half of 37
- an increase from 1 to 5 is more significant than an increase from 96 to 106 because the percent increase is much more substantial
- if $y = 3x + 2$ and I multiply an x value by 100 then the y value is almost, but not quite, also multiplied by 100

x	20	30	40	…	200	300	400
y	62	92	122		602	902	1202

($y = 3x + 2$ is pretty much the same as $y = 3x$ when x is big). When you realize that if you multiply x by 100, you almost, but not quite, multiply y by 100, you are using proportional reasoning.

Non-Examples

One container has 2 cups of water with 6 tsp. of sugar. Another container has 4 cups of water with 10 tsp. of sugar. Non-proportional thinking would say the second liquid is sweeter because it has more sugar.

When someone thinks that doubling the length and width of a rectangle doubles the area, it is non-proportional thinking.

$y = 2x + 3$ is not a proportional relationship (but for large values of x it is close to $y = 2x$ which is a proportional relationship commonly referred to as direct variation).

Appendix 2: Curriculum Connections Across the Grades: Primary

Kindergarten	Grade 1	Grade 2	Grade 3
• recognize some quantities without having to count, using a variety of tools or strategies (BIN 2, BIN 3) • demonstrate an understanding of number relationships for numbers from 0 to 10, through investigation (BIN 2) • investigate and develop strategies for composing and decomposing quantities to 10 (BIN 3, BIN 6)	• count forward by 1's, 2's, 5's, and 10's to 100, using a variety of tools and strategies (BIN 1, BIN 2, BIN 4) • count backwards from 20 by 2's and 5's, using a variety of tools (BIN1, BIN2, BIN4) • identify and describe various coins (i.e., penny, nickel, dime, quarter, $1 coin, $2 coin), using coin manipulatives or drawings, and state their value (BIN 1) • represent money amounts to 20¢, through investigation using coin manipulatives (BIN 1, BIN 3) • divide whole objects into parts and identify and describe, through investigation, equal-sized parts of the whole, using fractional names (BIN 3, BIN 4) • estimate the number of objects in a set, and check by counting (BIN 4) • compare two or three objects using measurable attributes and describe the objects using relative terms (BIM 2)	• count forward by 1's, 2's, 5's, 10's, and 25's to 200, using number lines and hundreds charts, starting from multiples of 1, 2, 5, and 10 (BIN 1, BIN 2) • count backwards by 1's from 50 and any number less than 50, and count backwards by 10's from 100 and any number less than 100, using number lines and hundreds charts (BIN 1, BIN 2) • locate whole numbers to 100 on a number line and on a partial number line (BIN 1, BIN 2, BIN 3, BIN 4) • compose and decompose two-digit numbers in a variety of ways, using concrete materials (BIN 3, BIN 5) • regroup fractional parts into wholes, using concrete materials (BIN 3) • determine, through investigation using concrete materials, the relationship between the number of fractional parts of a whole and the size of the fractional parts (BIN 3) • compare fractions using concrete materials, without using standard fractional notation (BIN 4) • represent and explain, through investigation using concrete materials and drawings, multiplication as the combining of equal groups (BIN 6) • represent and explain, through investigation using concrete materials and drawings, division as the sharing of a quantity equally (BIN 6) • represent a given growing or shrinking pattern in a variety of ways (BIA 2)	• count forward by 1's, 2's, 5's, 10's, and 100's to 1000 from various starting points, and by 25's to 1000 starting from multiples of 25, using a variety of tools and strategies (BIN 1, BIN 2, BIN 4) • count backwards by 2's, 5's, and 10's from 100 using multiples of 2, 5, and 10 as starting points, and count backwards by 100's from 1000 and any number less than 1000, using a variety of tools and strategies (BIN 1, BIN 2) • represent and explain, using concrete materials, the relationship among the numbers 1, 10, 100, and 1000 (BIN 1, BIN 2, BIN 5) • represent and describe the relationships between coins and bills up to $10 (BIN 1) • estimate, count, and represent (using the $ symbol) the value of a collection of coins and bills with a maximum value of $10 (BIN 1) • divide whole objects and sets of objects into equal parts, and identify the parts using fractional names, without using numbers in standard fractional notation (BIN 6) • relate multiplication of one-digit numbers and division by one-digit divisors to real- life situations, using a variety of tools and strategies (BIN 5) • multiply to 7 × 7and divide to 49 ÷ 7, using a variety of mental strategies (BIN 6)

Appendix 2: Curriculum Connections Across the Grades: Junior

Grade 4	Grade 5	Grade 6
• describe relationships that involve simple whole-number multiplication (BIN 3) • determine and explain, through investigation, the relationship between fractions (i.e., halves, fifths, tenths) and decimals to tenths, using a variety of tools and strategies (BIN 3) • demonstrate an understanding of simple multiplicative relationships involving unit rates, through investigation using concrete materials and drawings (BIN 4, BIN 5) • demonstrate an understanding of place value in whole numbers and decimal numbers from 0.1 to 10 000, using a variety of tools and strategies (BIN 1) • represent, compare, and order decimal numbers to tenths, using a variety of tools and using standard decimal notation (BIN 4) • count forward by halves, thirds, fourths, and tenths to beyond one whole, using concrete materials and number lines (BIN 3) • count forward by tenths from any decimal number expressed to one decimal place, using concrete materials and number lines (BIN 3) • divide two-digit whole numbers by one-digit whole numbers, using a variety of tools and student-generated algorithms (BIN 5, BIN 6)	• describe multiplicative relationships between quantities by using simple fractions and decimals (BIN 4) • determine and explain, through investigation using concrete materials, drawings, and calculators, the relationship between fractions (i.e., with denominators of 2, 4, 5, 10, 20, 25, 50, and 100) and their equivalent decimal forms (BIN 3) • demonstrate an understanding of simple multiplicative relationships involving whole-number rates, through investigation using concrete materials and drawings • demonstrate, through investigation, an understanding of variables as changing quantities, given equations with letters or other symbols that describe relationships involving simple rates (BIN 4) • demonstrate an understanding of place value in whole numbers and decimal numbers from 0.01 to 100 000, using a variety of tools and strategies (BIN 1) • represent, compare, and order whole numbers and decimal numbers from 0.01 to 100 000, using a variety of tools (BIN 1) • demonstrate and explain equivalent representations of a decimal number, using concrete materials and drawings (BIN 3) • demonstrate and explain the concept of equivalent fractions, using concrete materials (BIN 3) • multiply decimal numbers by 10, 100, 1000, and 10 000, and divide decimal numbers by 10 and 100, using mental strategies (BIN 6) • determine, through investigation using stacked congruent rectangular layers of concrete materials, the relationship between the height, the area of the base, and the volume of a rectangular prism, and generalize to develop the formula (i.e., *Volume = area of base × height*) (BIM 2)	• represent ratios found in real-life contexts, using concrete materials, drawings, and standard fractional notation (BIN 4) • determine and explain, through investigation using concrete materials, drawings, and calculators, the relationships among fractions (i.e., with denominators of 2, 4, 5, 10, 20, 25, 50, and 100), decimal numbers, and percents (BIN 3, BIN 4) • represent relationships using unit rates (BIN 4) • represent, compare, and order whole numbers and decimal numbers from 0.001 to 1 000 000, using a variety of tools (BIN 4) • demonstrate an understanding of place value in whole numbers and decimal numbers from 0.001 to 1 000 000, using a variety of tools and strategies (BIN 1) • estimate quantities using benchmarks of 10%, 25%, 50%, 75%, and 100% (BIN 2, BIN 4) • multiply whole numbers by 0.1, 0.01, and 0.001 using mental strategies (BIN 6) • multiply and divide decimal numbers by 10, 100, 1000, and 10 000 using mental strategies (BIN 6) • determine the term number of a given term in a growing pattern that is represented by a pattern rule in words, a table of values, or a graph (BIA 2) • determine, through investigation using a variety of tools and strategies, the relationship between the area of a rectangle and the areas of parallelograms and triangles, by decomposing and composing (BIM 2)

Appendix 2: Curriculum Connections Across the Grades: Intermediate

Grade 7	Grade 8
• select and justify the most appropriate representation of a quantity (i.e., fraction, decimal, percent) for a given context; • use estimation when solving problems involving operations with whole numbers, decimals, and percents, to help judge the reasonableness of a solution; • determine, through investigation, the relationships among fractions, decimals, percents, and ratios; • solve problems that involve determining whole number percents, using a variety of tools; • demonstrate an understanding of rate as a comparison, or ratio, of two measurements with different units; • select and justify the most appropriate representation of a quantity (i.e., fraction, decimal, percent) for a given context; • solve problems involving the calculation of unit rates; • solve problems that require conversion between metric units of area (i.e., square centimetres, square metres); • model real-life relationships involving constant rates where the initial condition starts at 0, through investigation using tables of values and graphs; • research and report on real-world applications of probabilities expressed in fraction, decimal, and percent form; • demonstrate an understanding that enlarging or reducing two-dimensional shapes creates similar shapes; • distinguish between and compare similar shapes and congruent shapes, using a variety of tools and strategies.	• represent, compare, and order rational numbers (i.e., positive and negative fractions and decimals to thousandths); • translate between equivalent forms of a number (i.e., decimals, fractions, percents); • solve problems involving percents expressed to one decimal place and whole-number percents greater than 100; • use estimation when solving problems involving operations with whole numbers, decimals, percents, integers, and fractions, to help judge the reasonableness of a solution; • identify and describe real-life situations involving two quantities that are directly proportional; • solve problems involving proportions, using concrete materials, drawings, and variables; • solve problems involving percent that arise from real-life contexts; • solve problems involving rates; • determine, through investigation using a variety of tools and strategies, the relationships for calculating the circumference and the area of a circle, and generalize to develop the formulas [i.e., Circumference of a circle = $\pi \times$ diameter; Area of a circle = $\pi \times (\text{radius})^2$]; • identify and describe trends, based on the rate of change of data from tables and graphs, using informal language; • compare two attributes or characteristics, using a variety of data management tools and strategies (i.e., pose a relevant question, then design an experiment or survey, collect and analyse the data, and draw conclusions); • compare, through investigation, the theoretical probability of an event (i.e., the ratio of the number of ways a favourable outcome can occur compared to the total number of possible outcomes) with experimental probability, and explain why they might differ; • determine, through investigation using a variety of tools, relationships among area, perimeter, corresponding side lengths, and corresponding angles of similar shapes; • compare two attributes or characteristics, using a scatter plot, and determine whether or not the scatter plot suggests a relationship.

Appendix 2: Curriculum Connections Across the Grades: Intermediate

Grade 9

Academic

- solve problems requiring the manipulation of expressions arising from applications of percent, ratio, rate, and proportion;
- solve problems that can be modelled with first-degree equations, and compare algebraic methods to other solution methods;
- identify, through investigation, some properties of linear relations (i.e., numerically, the first difference is a constant, which represents a constant rate of change; graphically, a straight line represents the relation), and apply these properties to determine whether a relation is linear or non-linear;
- determine values of a linear relation by using a table of values, by using the equation of the relation, and by interpolating or extrapolating from the graph of the relation;
- describe a situation that would explain the events illustrated by a given graph of a relationship between two variables;
- determine, through investigation, connections among the representations of a constant rate of change of a linear relation;
- identify, through investigation, properties of the slopes of lines and line segments, using graphing technology to facilitate investigations, where appropriate;
- describe the meaning of the slope and y-intercept for a linear relation arising from a realistic situation, and describe a situation that could be modelled by a given linear equation.

Applied

- illustrate equivalent ratios, using a variety of tools;
- represent, using equivalent ratios and proportions, directly proportional relationships arising from realistic situations;
- solve for the unknown value in a proportion, using a variety of methods;
- make comparisons using unit rates;
- solve problems involving ratios, rates, and directly proportional relationships in various contexts using a variety of methods;
- solve problems requiring the expression of percents, fractions, and decimals in their equivalent forms;
- simplify numerical expressions involving integers and rational numbers, with and without the use of technology;
- identify, through investigation, some properties of linear relations (i.e., numerically, the first difference is a constant, which represents a constant rate of change; graphically, a straight line represents the relation), and apply these properties to determine whether a relation is linear or non-linear;
- determine, through investigation, that the rate of change of a linear relation can be found by choosing any two points on the line that represents the relation, finding the vertical change between the points (i.e., the rise) and the horizontal change between the points (i.e., the run), and writing the ratio $\frac{rise}{run}$ (i.e., $\text{rate of change} = \frac{rise}{run}$);
- determine, through investigation, connections among the representations of a constant rate of change of a linear relation;
- express a linear relation as an equation in two variables, using the rate of change and the initial value;
- describe the meaning of the rate of change and the initial value for a linear relation arising from a realistic situation, and describe a situation that could be modelled by a given linear equation;
- determine values of a linear relation by using a table of values, by using the equation of the relation, and by interpolating or extrapolating from the graph of the relation;
- describe a situation that would explain the events illustrated by a given graph of a relationship between two variables;
- solve problems that can be modelled with first-degree equations, and compare the algebraic method to other solution methods.

Grade 10

Academic

- solve problems that arise from realistic situations described in words or represented by linear systems of two equations involving two variables, by choosing an appropriate algebraic or graphical method;
- determine, through investigation, the relationship between the ratio of two sides in a right triangle and the ratio of the two corresponding sides in a similar right triangle, and define the sine, cosine, and tangent ratios;
- determine the measures of the sides and angles in right triangles, using the primary trigonometric ratios and the Pythagorean theorem;
- solve problems involving the measures of sides and angles in right triangles in real-life applications, using the primary trigonometric ratios and the Pythagorean theorem;
- explore the development of the sine law within acute triangles;
- explore the development of the cosine law within acute triangles;
- determine, through investigation, some characteristics and properties of geometric figures;
- verify, through investigation, the properties of similar triangles;
- describe and compare the concepts of similarity and congruence;
- solve problems involving similar triangles in realistic situations.

Applied

- verify, through investigation, properties of similar triangles;
- determine the lengths of sides of similar triangles, using proportional reasoning;
- determine, through investigation, the relationship between the ratio of two sides in a right triangle and the ratio of the two corresponding sides in a similar right triangle, and define the sine, cosine, and tangent ratios;
- determine the measures of the sides and angles in right triangles, using the primary trigonometric ratios and the Pythagorean theorem;
- solve problems involving the measures of sides and angles in right triangles in real-life applications, using the primary trigonometric ratios and the Pythagorean theorem;
- connect the rate of change of a linear relation to the slope of the line, and define the slope as the ratio $m = \frac{rise}{run}$.

Appendix 2: Curriculum Connections Across the Grades: Senior

Grade 11			
MCR3U Functions (University)	MCF3M Functions and Applications (University/College)	MBF3C Foundations for College Mathematics (College)	MEL3E Mathematics for Work and Everyday Life (Workplace)
• simplify polynomial expressions by adding, subtracting, and multilplying; • verify, through investigation with and without technology, that $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, $a \geq 0, b \geq 0$, and use this relationship to simplify radicals and expressions obtained by adding, subtracting and multiplying; • simplify rational expressions by adding, subtracting, multiplying, and dividing, and state the restrictions on the variable values; • determine if two given algebraic expressions are equivalent (i.,e., by simplifying; by substituting values); • determine, through investigation using a variety of tools and strategies, the value of a power with a rational exponent (i.e., $x^{\frac{m}{n}}$ where $x > 0$ and m and n are integers); • simplify algebraic expressions containing integer and rational exponents, and evaluate numeric expressions containing integer and rational exponents and rational bases; • distinguish exponential functions from linear and quadratic functions by making comparisons in a variety of ways; • identify sequences as arithmetic, geometric, or neither, given a numeric or algebraic representation; • determine the exact values of the sine, cosine, and tangent of the special angles: 0°, 30°, 45°, 60°, and 90°; • pose problems involving right triangles and oblique triangles in two dimensional settings, and solve these and other such problems using the primary trigonometric ratios, the cosine law and the sine law (including the ambiguous case).	• solve problems, including those that arise from real-world applications, by determining the measures of the sides and angles of right triangles using the primary trigonometric ratios; • solve problems involving two right triangles in two dimensions; • verify, through investigation using technology, the sine law and the cosine law; • solve probems that require the use of the sine law or the cosine law in acute triangles, including problems arising from real-world applications; • determine, through investigation using a variety of tools and strategies, the value of a power with a rational exponent (i.e., $x^{\frac{m}{n}}$ where $x > 0$ and m and n are integers); • evaluate, with and without technology, numerical expressions containing integer and rational expnents and rational bases; • determine, through investigation, the exponent rules for multipying and dividing numeric expressions involving expontents, and the exponent rule for simplifying numerical expressions involving a power of a power, and use the rules to simplify numerical expressions containing integer exponents; • distinguish exponential functions from linear and quadratic functions by making comparisons in a variety of ways, within the same context when possible; • compare, using a table of values and graphs, the simple and compound interest earned for a given principal and a fixed interest rate over time; • solve problems, including those that arise from real-world applications, by determining the measures of the sides and angles of right triangles using the primary trigonometric ratios; • verify, through investigation using technology the sine law and the cosine law.	• determine, through investigation using a variety of tools and strategies, and describe the meaning of negavitve exponents and of zero as an exponent; • evaluate, with and without technology, numeric expressions containing integer exponents and rational bases; • determine, through investigation, the exponent rules for mulitpying and dividing numerical expressions involving exponents, and the exponent rule for simplifying numerical expressions involving a power of a power; • distinguish exponential relations from linear and quadratic relations by making comparisons in a variety of ways, within the same context when possible; • determine, through investigation using technology, the effect on the future value of a compound interest investment or loan of changing the total length of time, the interest rate, or the compounding period; • determine the theoretical probability of an event (i.e., the ratio of the number of favourable outcomes to the total number of possible outcomes, where all outcomes are equally likely), and represent the probability in a variety of ways; • represent three-dimensional objects, using concrete materials and design or drawing software, in a variety of ways; • solve problems, including those that arise from real-world applications by determining the measures of the sides and angles of right triangles using primary trigonometric ratios.	• estimate the sale price before taxes when making a purchase; • compare the unit prices of related items to help determine the best buy; • determine, through investigation using technology, the effect on simple interest of changes in the principal, interest rate, or time, and solve problems involving applications of simple interest; • determine, through investigation using technology, the effect on the future value of a compound interest investment of changing the total length of time, the interest rate, or the compounding period; • determine distances represented on maps, using given scales.

Appendix 2: Curriculum Connections Across the Grades: Senior

<table>
<tr><th colspan="3">Grade 12</th></tr>
<tr>
<td>MHF4U Advanced Functions
(University)
• recognize that the rate of change for a function is a comparison of changes in the dependent variable to changes in the independent variable, and distinguish situations in which the rate of change is zero, constant, or changing by examining applications, including those arising from real-world situations;
• compare, through investigation using a variety of tools and strategies the characteristics of various functions;
• recognize the radian as an alternative unit to the degree for angle measurement, define the radian measure of an angle as the length of the arc that subtends this angle at the centre of a unit circle, and develop and apply the relationship between radian and degree measure;
• determine, without technology, the exact values of the primary trigonometric ratios and the reciprocal trigonometric ratios for the special angles $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples less than or equal to 2π;
• recognize that trigonometric identities are equations that are true for every value in the domain (i.e., a counter-example can be used to show that an equation is not an identity), prove trigonometric identities through the application of reasoning skills, using a variety of relationships, and verify identities using technology.</td>
<td>MDM4U Mathematics of Data Management
(University)
• determine the theoretical probability, π (i.e., a value from 0 to 1) of each outcome of a discrete sample space, recognize that the sum of the probabilities of the outcomes is 1, (i.e., for n outcomes, $P_1 + P_2 + P_3 + \ldots + P_n = 1$) recognize that the probabilities π form the probability distribution associated with the sample space, and solve related problems;
• solve simple problems using techniques for counting permutations and combinations, where all objects are distinct, and express the solutions using standard combinatorial notation;
• solve introductory counting problems involving the additive counting principle and the multiplicative counting principle;
• interpret statistics presented in the media, and explain how the media, the advertising industry, and others use and misuse statistics to promote a certain point of view;
• assess the validity of conclusions presented in the media by examining sources of data, including Internet sources, methods of data collection, and possible sources of bias, and by questioning the analysis of the data and conclusions drawn from the data.</td>
<td>MAP4C Foundations for College Mathematics
(College)
• recognize that a linear model corresponds to a constant increase or decrease over equal intervals and that an exponential model corresponds to a constant percentage increase or decrease over equal intervals, select a model (i.e., linear, quadratic, exponential) to represent the relationship between numerical data graphically and algebraically, using a variety of tools and strategies, and solve related problems;
• identify when the rate of change is zero, constant or changing, given a table of values or a graph of a relations, and compare two graphs by describing rate of change;
• perform required conversions between the imperial systems and the metric systems using a variety of tools, as necessary within applications;
• solve problems in two dimensions using metric or imperial measurements, including problems that arise from real-world applications by determine the measures of the sides and angles of right triangles using the primary trigonometric ratios, and of acute triangles using the sine law and the cosine law;
• determine an algebraic summary of the relationship between two variables that appear to be linearly related, using a variety of tools and strategies and solve related problems;
• describe possible interpretations of the line of best fit of a scatter plot and reasons for misinterpretations.</td>
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<tr>
<td>MCV4U Calculus and Vectors
(University)
• determine, through investigation using technology, the graph of the derivative $f'(x)$ or $\frac{dy}{dx}$ of a given exponential function [i.e., $f(x) = a^x > 0, a \neq 1$)] and make connections between the graphs of $f(x)$ and $f'(x)$ or y and $\frac{dy}{dx}$;
• verify, using technology, that the derivative of the exponential function $f(x) = a^x$ is $f'(x) = a^x \ln a$ for various values of a;
• represent a vector in two-space geometrically as a directed line segment, with directions expressed in different ways, and algebraically, and recognize vectors with the same magnitude and direction but different positions as equal vectors;
• perform the operations of addition, subtraction, and scalar multiplication on vectors represented as directed line segments in two-space, and on vectors represented in Cartesian form in two-space and three-space.</td>
<td>MCT4C Mathematics for College Technology
(College)
• determine the exact values of the sine, cosine, and tangent of the special angles: 0°, 30°, 45°, 60°, and 90° and their multiples;
• solve multi-step problems in two and three dimensions, including those that arise from real-world applications, by determining the measures of the sides and angles of right triangles using the primary trigonometric ratios;
• perform required conversions between the imperial system and the metric system using a variety of tools as necessary within applications.</td>
<td>MEL4E Mathematics for Work and Everyday Life
(Workplace)
• determine the theoretical probability of an event (i.e., the ratio of the number of favourable outcomes to the total number of possible outcomes, where all outcomes are equally likely), and represent the probability in a variety of ways;
• convert measures within systems, as required within applications that arise from familiar contexts;
• identify and describe applications of ratio and rate, and recognize and represent equivalent ratios and equivalent rates, using a variety of tools;
• identify situations in which it is useful to make comparisons using unit rates, and solve problems that involve comparisons of unit rates;
• identify and describe real-world applications of proportional reasoning, distinguish between a situation involving a proportional relationship and a situation involving a non-proportional relationship in a personal and/or workplace context, and explain their reasoning;
• identify and describe the possible consequences of errors in proportional reasoning.</td>
</tr>
</table>

Glossary

Unitizing is the place value understanding that ten can be thought of as one group of ten or ten individual units. When you are counting and you pass 9, the 10 means 1 group of 10 and then 11 means 1 ten and 1 one, and 12 means 1 ten and 2 ones, and so on.

Composing is making or creating a whole by putting together its basic parts.

Decomposing is separating a whole into its basic parts.

Skip Counting is a mathematics technique taught as a kind of multiplication. To skip count is to recite the multiples of a number (e.g., 3, 6, 9, 12, 15, and so on).

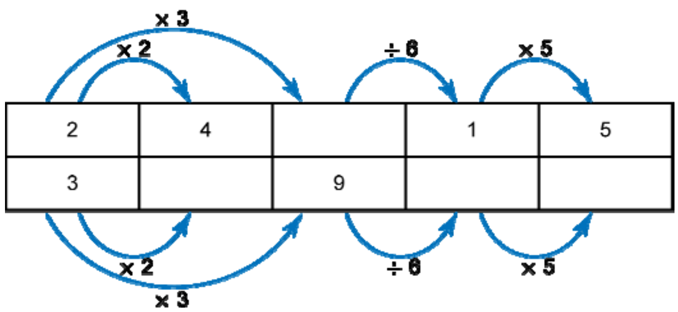
Ratios are comparisons of two or more values with the same units (e.g., 2:5 might be the ratio of the number of blue balls to red balls if a set has 2 blue balls and 5 red ones OR 1:100 might be the scale ratio of a map where 1 cm represents 1 m).

2:1

2 hands and feet and arms and legs per 1 person

(A student might think of this combination of a picture and words as a definition. We should encourage this sort of personal definition.)

Using a Ratio Table is a strategy to show equivalent ratios. Entries in a column are multiplied or divided by the same amount. This ratio table shows the proportion 2:3 = 4:6 = 6:9 = 1:1.5 = 5:7.5. A ratio entry of 1 is useful for calculations where the multiplication or division is not obvious.



Percents are special ratios where the second term is 100.

Rates are comparisons of two values with different units (e.g., 50¢/cookie, 25 km/h) Note: Not all mathematicians agree with this definition—some do not distinguish between ratios and rates.

A **proportion** is a statement about the equivalence of two ratios (e.g., 2:3 = 4:6). Proportions can be solved mechanically without doing any proportional thinking.

Proportional describes the relationship between two variables when the values of one variable are always a constant multiple of the corresponding values of the other. (In Grade 9, this is called direct variation.)

Multiplicative describes a relationship that involves multiplication.

Additive describes a relationship involving addition.

Absolute change measures change between one amount and another.

Relative change measures change of one amount to another amount in relation to a change between another two amounts.

Similarity describes a relationship between two geometric objects in which the two objects are congruent as a result of the uniform enlarging or shrinking of one of them.

Resources

Manipulatives

- coloured tiles
- cubes/geoboards
- fraction circles/rings/rods
- grid paper
- pattern blocks
- coins
- base 10 blocks

Technology

- The Geometer's Sketchpad 4®
- Fathom
- calculators/graphing calculators
- spreadsheet software
- CLIPS – Critical Learning Instructional Paths Supports www.mathclips.ca.
  Interactive learning activities and resources developed by the Ontario Ministry of Education That provide precise, personalized learning activities for instruction, remediation, or enrichment.
  These CLIPS support Proportional Reasoning:
  - Fractions Representing Part/Whole Relationships
  - Representations of Linear Growing Patterns
  - Periodic and Sine Functions and Their Transformations (CLIP 3 – The Sine Ratio)
- Virtual manipulatives: National Library of Virtual Manipulatives
  http://nlvm.usu.edu/en/nav/vlibrary.html

Web Resources

- Annenberg Media Learning Math

  Rational Numbers and Proportional Reasoning: Primary/Junior

  http://www.learner.org/courses/learningmath/number/session8/index.html
  "Begin examining rational numbers. Explore a model for computations with fractions. Analyze proportional reasoning and the difference between absolute and relative thinking. Explore ways to represent proportional relationships and the resulting operations with ratios. Examine how ratios can represent either part-part or part-whole comparisons, depending on how you define the unit, and explore how this affects their behavior in computations."

  Fractions, Percents, and Ratios: Primary/Junior

  http://www.learner.org/courses/learningmath/number/session9/index.html
  "Continue exploring rational numbers, working with an area model for multiplication and division with fractions, and examining operations with decimals. Explore percents and the relationships among representations using fractions, decimals, and percents. Examine benchmarks for understanding percents, especially percents less than 10 and greater than 100. Consider ways to use an elastic model, an area model, and other models to discuss percents. Explore some ratios that occur in nature."

  Proportional Reasoning: Junior/Intermediate

  http://www.learner.org/courses/learningmath/algebra/session4/index.html
  "Look at direct variation and proportional reasoning. This investigation will help you to differentiate between relative and absolute meanings of "more" and to compare ratios without using common denominator algorithms. Topics include differentiating between additive and multiplicative processes and their effects on scale and proportionality, and interpreting graphs that represent proportional relationships or direct variation."

Linear Functions and Slope: Intermediate
http://www.learner.org/courses/learningmath/algebra/session5/index.html
"Explore linear relationships by looking at lines and slopes. Using computer spreadsheets, examine dynamic dependence and linear relationships and learn to recognize linear relationships expressed in tables, equations, and graphs. Also, explore the role of slope and dependent and independent variables in graphs of linear relationships, and the relationship of rates to slopes and equations."

- Continuum and Connections: Grade 7 through Grade 10 Proportional Reasoning
  http://www.edugains.ca/newsite/math/continuumconnections.htm
  This content-based package includes problems with solutions connected to the Mathematical Processes and makes connections across the grades for linking with other strands, with instruction, and with the problems.

- DI Package Enhancements 2008: Math DI Cards
  http://www.edugains.ca/newsite/di/dimathcards.htm
  Mathematical Process links include Sample Questions and Sample Feedback for each Process.

- GAINS Classroom Practice
  http://www.edugains.ca/newsite/math/toolsandstrategies.htm
  Elastic Metre Manipulative, Grade 9 Applied (Video)
  Fermi Problems (Adobe PDF)

- Geometer's Sketchpad Supports for Proportional Reasoning

  Geometer's Sketchpad is licensed for use by Ontario teachers and students. These are links to sketches that can be used to develop proportional reasoning.

  The Geometer's Sketchpad® Resource Center: Primary/Junior

  http://www.dynamicgeometry.com/General_Resources/Classroom_Activities/KCPT/Activities_for_Young_Learners/Sketchpad_for_Grades_3-5/Activities.html
  1) Grouping: Explore different-sized groupings for a fixed number of circles.
  2) Jump Along: Jump along a number line using different numbers of jumps and jump sizes.
  3) Measuring Fish: Measure the size of fish with "binkers."
  4) Similarity: Explore similarity and congruence. Create your own re-sizeable logo.

  Prism-Neo: Properties of 2D Shapes: Intermediate

  http://www.nearnorthschools.ca/it/PRISMNEO/list.asp?concept=Properties%20of%202D%20Shapes&grade=All
  Circles and Pi: "Unroll" a circle to investigate the proportional relationship between a diagonal and the circumference.
  Relating Circumference and Diameter: Change the size of a circle and observe changes in the measurements of its radius, diameter, and circumference, as well as the ratio C:d.

  Prism-Neo: Ratio and Relationships: Intermediate

  http://www.nearnorthschools.ca/it/PRISMNEO/list.asp?concept=Ratios&grade=All
  Ratio Relationships: Explore similar rectangles and ratios as scales then try a "guestimating" bug size activity.

  Prism-Neo: Transformations: Intermediate

  http://www.nearnorthschools.ca/it/PRISMNEO/list.asp?concept=Transformations&grade=All
  Dilatations – what a stretch! Explore properties of enlargements and reductions.

  Ministry of Education: Senior Grade 12 GSP, MCV4U

  http://www.edu.gov.on.ca/eng/studentsuccess/lms/files/GSP/grade12gsp.html
  Secant Slope: Explore the slope of a secant.

  Ministry of Education: Senior Grade 11 GSP Exponentials

  http://www.edu.gov.on.ca/eng/studentsuccess/lms/files/GSP/grade11gsp.html
  Simple and Compound Interest

- Think Literacy: Subject-Specific Documents Grades 7-12, Mathematics
  http://www.edu.gov.on.ca/eng/studentsuccess/thinkliteracy/library.html#subjects
  Mathematics, Grades 7-9 (2004) (PDF, 4.27 MB)
  The Frayer model (pp. 38–42).

- University of Maryland Fermi Problems Site
  http://www.physics.umd.edu/perg/fermi/fermi.htm
  Fermi problems involve estimation with large (or really small) numbers and most Fermi problems involve proportional thinking. This site has an extensive collection of Fermi problems under a General category that could be used in a variety of grades. In addition, there are Fermi problems, categorized under Science topics, some of which could be used for Senior Math.

Print Resources

Big Ideas from Dr. Small, Grades 4-8; Marian Small, Nelson Education, 2009.
ISBN 978-0-17-635713-9

Big Ideas from Dr. Small, Grades 9-12; Marian Small, Amy Lin; Nelson Education, 2011.
ISBN 978-0-17-650351-2

Classroom Activities for Making Sense of Fractions, Ratios, and Proportions, (NCTM) Yearbook 2002.
ISBN 0-87353-519-7

Good Questions: Great Ways to Differentiate Mathematics Instruction, Marian Small, Nelson Education, 2009. ISBN 978-0-8077-4978-4

Making Sense of Fractions, Ratios, and Proportions, 2002 Yearbook (NCTM). ISBN 0-87353-519-7

Mathematics Teaching Cases Facilitator's Discussion Guide: Fractions, Decimals, Ratios, & Percents (C. Barnett, D. Goldstein, B. Jackson) Heinemann, 1994. ISBN 0-435-08358-9

Mathematics Teaching Cases: Fractions, Decimals, Ratios, & Percents (C. Barnett, D. Goldstein, B. Jackson) Heinemann, 1994. ISBN 0-435-08357-0

More Good Questions: Great Ways to Differentiate Secondary Mathematics Instruction (Marian Small, Amy Lin) Nelson Education, 2010. ISBN 978-0-8077-5088-9

Proportional Reasoning: Algebraic Thinking Series, AIMS Activities Grades 6-9, Spectrum, 2003.
ISBN 1-881431-78-9

Teaching Fractions and Ratios for Understanding. (Susan Lamon) Lawrence Erlbaum Associates, Publishers, 1999. ISBN 0-8058-2940-7 (See More In-Depth Discussion of the Reasoning Activities.)

Teaching Student-Centered Mathematics, K-3, (J. Van de Walle) Pearson, 2006.
ISBN 0-205-40843-5

Teaching Student-Centered Mathematics, 3-5, (J. Van de Walle) Pearson, 2006.
ISBN 0-205-40844-3

Teaching Student-Centered Mathematics, 5-8, (J. Van de Walle) Pearson, 2006.
ISBN 0-205-41797-3

Articles

Are We Golden? Investigations with the Golden Ratio—MTMS Article
http://www.nctm.org/eresources/article_summary.asp?from=B&uri=MTMS2007-10-150a

Conceptualizing Ratios with Look-Alike Polygons—MTMS Article
http://www.nctm.org/eresources/article_summary.asp?from=B&uri=MTMS2003-04-426a

Crossing the Bridge to Formal Proportional Reasoning—MTMS Article
http://www.nctm.org/eresources/article_summary.asp?from=B&uri=MTMS2003-04-420a

Exploring Proportional reasoning Through Movies and literature (C. Beckmann) in NCTM Mathematics Teaching in the Middle School January 2004.

Figuring Fitness: Nutrition in the Middle Grades (C. Beckmann, E. Billings) in NCTM Mathematics Teaching in the Middle School August 2004.

From the Classroom Student constructed problems… – TCM article
http://www.nctm.org/eresources/article_summary.asp?from=B&uri=TCM2010-08-16a

Improving Middle School Teachers' Reasoning about Proportional Reasoning—MTMS Article
http://www.nctm.org/eresources/article_summary.asp?from=B&uri=MTMS2003-04-398a

Investigating Mathematical Thinking and Discourse with Ratio Triplets—MTMS Article
http://www.nctm.org/eresources/article_summary.asp?from=B&uri=MTMS2008-08-12a

Learning and Teaching Ratio and Proportion: Research Implications (K. Cramer, T. Post, S. Currier) 1993
http://www.cehd.umn.edu/rationalnumberproject/93_4.html

NCTM Focus Issue: Proportional Reasoning, Mathematics Teaching in the Middle School April 2003.

Proportional Reasoning—MTMS Article
http://www.nctm.org/eresources/article_summary.asp?from=B&uri=MTMS2000-01-310a

Proportionality: A Unifying Theme for the Middle Grades—MTMS Article

http://www.nctm.org/eresources/article_summary.asp?from=B&uri=MTMS2003-04-392a

Reflection Guides

http://www.nctm.org/profdev/content.aspx?id=23504

http://www.nctm.org/eresources/article_summary.asp?URI=MTMS2003-04-392a&from=B

Problems that Encourage Proportional Sense

http://www.nctm.org/eresources/view_media.asp?article_id=1537

A Science Application of Area and Ratio Concepts (V. Horak) NCTM Mathematics Teaching in the Middle School April 2006.

Using a Pattern Table to Solve Contextualized Proportion Problems
http://www.nctm.org/eresources/article_summary.asp?from=B&uri=MTMS2003-04-432a

Whodunit? Exloring Proprotional Reasoning Through the Footprint Problem (K. Koellner-Clark, R. Lesh) in School Science and Mathematics, February 2003.

Picture Books

Counting on Frank. (Clement, R.) Gareth Stevens, 1991. ISBN 0-8368-0358-2

How Much is a Million? (Schwartz, D.) Lothrop, Lee & Shepard Books, 1985. ISBN 0-688-04049-7

If You Hopped Like a Frog. (Schwartz, D.) Scholastic Press, 1999. ISBN 0-590-09857-8