GAP CLOSING

2D Measurement

Intermeditate / Senior
Facilitator's Guide

2-D Measurement

The Facilitator's Guide for this entire module, including the Diagnostic plus all Intervention Materials, can be downloaded at http://www.edugains.ca/resources/LearningMaterials/GapClosing/Grade9/8-2DMeasurement_FG_IS.pdf. The Student Book for this module can be downloaded at http://www.edugains.ca/resources/LearningMaterials/GapClosing/Grade9/8-2Dmeasurement_SB_IS.pdf.

2-D MEASUREMENT

Relevant Expectations for Grade 9

MPM1D

Measurement and Geometry

- solve problems involving the areas and perimeters of composite two-dimensional shapes
- develop, through investigation, the formulas for the volume of a pyramid, a cone, and a sphere
- determine, through investigation, the relationship for calculating the surface area of a pyramid
- solve problems involving the surfaces areas and volumes of prisms, pyramids, cylinders, cones, and spheres, including composite figures

MPM1P

Measurement and Geometry

- solve problems involving the areas and perimeters of composite two-dimensional shapes
- develop, through investigation, the formulas for the volume of a pyramid, a cone, and a sphere
- solve problems involving the volumes of prisms, pyramids, cylinders, cones, and spheres

Possible reasons why a student may struggle with areas and perimeters of 2-D shapes, including circles.

Students may struggle with calculating areas of parallelograms, triangles, trapezoids, circles and composite shapes or circumferences of circles.

Some of the problems include:
- using the adjacent sides of a parallelogram rather than a base and height to determine its area
- taking half of both the height and base to calculate the area of a triangle
- multiplying the height by either only one base or the product of the two bases rather than half the sum of the two bases to determine the area of a trapezoid
- using a third side length, rather than a height, to determine the area of a trapezoid
- an inability to visualize how to decompose a shape into simpler shapes to calculate its area
- difficulty deducing information to indirectly determine necessary measurements of a composite shape
- mixing up the radius and diameter in formulas for the area and circumference of a circle
- lack of awareness that it is sometimes useful to subtract the area of one shape from the area of another to determine the area of a particular shape
- inability to apply the measurement formulas in more complex situations

DIAGNOSTIC

Administer the diagnostic

Using diagnostic results to personalize interventions

Materials
- coloured pencils, markers
- Formula Sheet
- calculators (optional)

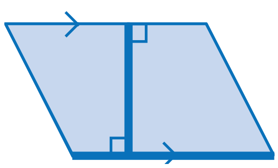
Intervention materials are included on each of these topics:
- areas of parallelograms, triangles, and trapezoids
- areas of composite shapes
- circumference and areas of circles

You may use all or only part of these sets of materials, based on student performance with the diagnostic. If students need help in understanding the intent of a question in the diagnostic, you are encouraged to clarify that intent.

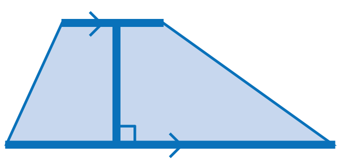
Evaluating Diagnostic Results	Suggested Intervention Materials
If students struggle with Questions 1–4	use *Areas of Parallelograms, Triangles, and Trapezoids*
If students struggle with Questions 5–7	use *Areas of Composite Shapes*
If students struggle with Questions 8–11	use *Circumference and Areas of Circles*

Solutions

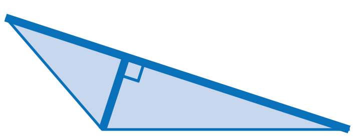
1. **a)** e.g.,



b) e.g.,



c) e.g.,



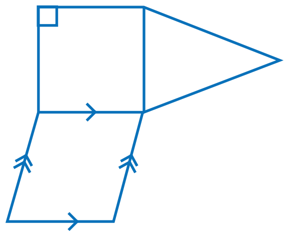
2. **a)** 9 square units **b)** 20 square units **c)** 42 square units

3. 5 cm

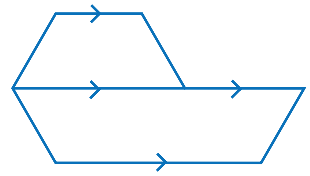
4. e.g., height of 4 cm and bases of 3 cm and 7 cm

5. e.g.,

a)



b)



6. **a)** The rectangle on the left has area of 40 cm^2
The triangle on the right has a base of 10 – 3 = 7 cm
and height of 10 – 4 = 6 cm; area = 21 cm^2
Total area is 61 cm^2.

b) The trapezoid in the middle has area of $4 \times \frac{(3 + 5)}{2} = 16$ cm^2.
The triangle at left has area of $\frac{5}{2}$ cm^2.
The triangle at right has a base of 3 cm and height of (6 – 4) = 2 cm;
area is 3 cm^2.
Total area is 21.5 cm^2

7. **a)** 15 cm^2 **b)** 10 cm^2

[**Note:** For questions involving π, answers are provided as well as using the number π using an estimate of 3.14.]

8. **a)** 25 π cm^2 (or 78.5 cm^2) **b)** 10 π cm (or 31.4 cm)

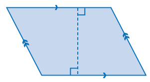
9. e.g., $\pi d = 4 \times 3 = 12$, so $d = \frac{12}{\pi}$ (or 3.8 cm)

10. b

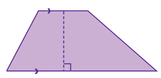
11. a, c, e

Diagnostic

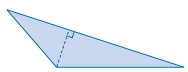
1. Use a marker or a coloured pencil to indicate the lengths needed to calculate the area of each shape. Mark only the necessary measurements on each shape.
   a) Parallelogram



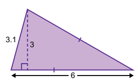
   b) Trapezoid



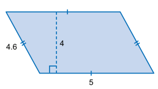
   c) Triangle



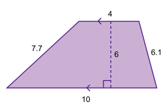
2. Calculate each area:
   a)



   b)



   c)

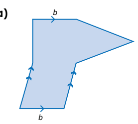


Diagnostic (Continued)

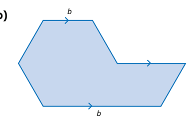
3. A triangle has a base of 5 cm and a height of 10 cm. A parallelogram with a base of 5 cm has the same area. What is the height of the parallelogram?

4. A trapezoid has an area of 20 cm². What could the height and base lengths be?

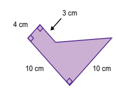
5. Show a way to divide each shape into any combination of triangles, rectangles, parallelograms, or trapezoids.
   a)



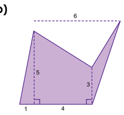
   b)



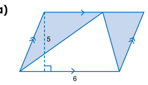
6. Calculate the area of each shape. Show your work.
   a)



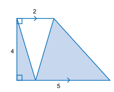
   b)



7. What is the area of the shaded space?
   a)

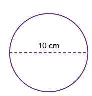


   b)



Diagnostic (Continued)

8. The diameter of a circle is 10 cm.
   a) What is the area of the circle?



   b) What is the circumference of the circle?

9. The circumference of a circle is about the same as the perimeter of a square with side length 3 cm. What is the diameter of the circle? Show your thinking.

10. A string is three times the length of the diameter of a circle. Circle the statement that is true:
   A That string would go about halfway around the circle.
   B That string would go almost all the way around the circle.
   C That string would go around the circle and a bit more.
   D That string would almost go around the circle twice.

11. The area of a circle is about 20 cm². Circle all of the statements that are true.
   a) The radius is about 2.5 cm.
   b) The radius is about 5 cm.
   c) The diameter is about 5 cm.
   d) The diameter is about 10 cm.
   e) The circumference is about 15 cm.
   f) The circumference is about 30 cm.

USING INTERVENTION MATERIALS

The purpose of the suggested work is to help students build a foundation for successfully working with volumes and surface areas of 3-D figures, as well as working with the areas of more complex 2-D shapes.

Each set of intervention materials includes a single-task Open Question approach and a multiple-question Think Sheet approach. These approaches both address the same learning goals, and represent different ways of engaging and interacting with learners. You could assign just one of these approaches, or sequence the Open Question approach before, or after the Think Sheet approach.

Suggestions are provided for how best to facilitate learning before, during, and after using your choice of approaches. This three-part structure consists of:

- Questions to ask before using the approach
- Using the approach
- Consolidating and reflecting on the approach

Areas of Parallelograms, Triangles, and Trapezoids

Learning Goal

- recognizing the efficiency of using formulas to simplify area calculations.

Materials
- rulers
- Formula Sheet
- Rectangle and Parallelogram template
- calculators (optional)

Open Question

Questions to Ask Before Using the Open Question

◊ *If I drew a rectangle, what would you want to know about it to figure out its area?* (e.g., the length and width)
◊ *If you knew the length and width, would that be enough information to figure out the area without seeing the shape?* (Yes.)
◊ *How is a parallelogram different from a rectangle?* (e.g., A parallelogram might be slanted.)

Show a rectangle that is 5 cm × 10 cm and also show a very "flat" parallelogram with the same side lengths. (Rectangle and Parallelogram template.)

◊ *If a parallelogram that is not a rectangle and a rectangle have all the same side lengths, could they have the same area?* (e.g., It does not look like it. The rectangle looks like it has more area.)

Using the Open Question

Students can choose whole number or rational number areas.

By viewing or listening to student responses, note if they:
- are able to apply formulas to determine the relationship between area and linear dimensions for triangles, parallelograms, and trapezoids;
- realize that the height of a triangle that has the same area and same base as a given parallelogram is twice the height of that parallelogram;
- realize that all three shapes could have the same height and area, but different base lengths;
- notice that if the base of a triangle or parallelogram with a given area increases, the height must decrease;
- realize that the area can be divided by the height to determine the base of a parallelogram or the average of the two bases of a trapezoid.

Consolidating and Reflecting on the Open Question

◊ *What area did you select first?* (e.g., 100 cm^2)
◊ *How did you choose dimensions for a parallelogram with that area?* (e.g., I thought of two numbers that multiply to 100. One number is the base and one number is the height.)
◊ *What did you do differently for the triangle?* (e.g., I used the same base but a height that was twice as much.) *Why did you do that?* (e.g., You take half of the product to get the area of a triangle.)
◊ *What did you do differently for the trapezoid?* (e.g., I used the same height of 10 as I did for the parallelogram, but I needed two bases, so instead of using 10, I used 6 and 14, so one was longer than 10 and one was the same amount shorter than 10.)
◊ *Why did you pick numbers like 100, 50, and 40 for your areas?* (e.g., They have easy factors and I needed those factors for bases and heights.)
◊ *What would you do differently if the area had been a more challenging number, for example, 35.2?* (e.g., I would pick any number I wanted for the height and would have to do some divisions with my calculator to figure out the bases.)

Solutions

e.g.,

Step 1

Area = 100 cm^2

Parallelogram: base of 10 cm, height of 10 cm

Triangle: base of 20 cm; height of 10 cm

Trapezoid: height of 10 cm, bases of 8 cm and 12 cm

Step 2

Parallelogram: base of 50 cm, height of 2 cm

Triangle: base of 50 cm; height of 4 cm

Trapezoid: height of 5 cm, bases of 16 cm and 24 cm

Step 3

Area = 50 cm^2

Parallelogram: base of 10 cm, height of 5 cm

Triangle: base of 10 cm; height of 10 cm

Trapezoid: height of 5 cm, bases of 8 cm and 12 cm

Parallelogram: base of 25 cm, height of 2 cm
Triangle: base of 25 cm; height of 4 cm
Trapezoid: height of 10 cm, bases of 3 cm and 7 cm

Area = 40 cm^2

Parallelogram: base of 10 cm, height of 4 cm
Triangle: base of 10 cm; height of 8 cm
Trapezoid: height of 5 cm, bases of 6 cm and 10 cm

Parallelogram: base of 8 cm, height of 5 cm
Triangle: base of 8 cm; height of 10 cm
Trapezoid: height of 8 cm, bases of 3 cm and 7 cm

Think Sheet

Materials
- rulers
- Formula Sheet
- Rectangle and Parallelogram template
- calculators (optional)

Questions to Ask Before Assigning the Think Sheet

◊ *If I drew a rectangle, what would you need to know about it to figure out its area?* (e.g., the length and width.)
◊ *If you knew the length and width, would that be enough information to figure out the area without seeing the shape?* (Yes.)
◊ *How is a parallelogram different from a rectangle?* (e.g., A parallelogram might be slanted.)

Show a rectangle that is 5 cm × 10 cm and also show a very “flat” parallelogram with the same side lengths. (Rectangle and Parallelogram template)

◊ *If a parallelogram that is not a rectangle and a rectangle have all the same side lengths, could they have the same area?* (e.g., It does not look like it. The rectangle looks like it has more area.)

Using the Think Sheet

Read through the introductory box with the students and make sure they understand the material explained in the instructional box.

Provide the Formula Sheet template and suggest that they use their rulers. Some students might also use calculators for some questions.

Ensure that students can identify trapezoids and parallelograms.

Assign the tasks.

By viewing or listening to student responses, note if they:
- are able to apply formulas to determine the relationship between area and linear dimensions for triangles, parallelograms, and trapezoids, whether the measurements are provided or need to be determined;
- recognize which linear measurements are needed, and which are not, to determine an area;
- can determine a height of a triangle, parallelogram, or trapezoid given the area and lengths of one or both bases;
- recognize that different base/height combinations can be used to determine the area of a triangle but that they all result in the same value;
- realize that the height of a triangle with the same area and the same base as a given parallelogram is twice the height of the parallelogram;
- recognize that different shapes can have the same area;
- recognize that a triangle with half the height and half the base of another has one-fourth the area;
- can develop the formula for the area of a trapezoid using knowledge of the formula for the area of a triangle.

Consolidating and Reflecting: Questions to Ask After Using the Think Sheet
◊ *Why did you only use two of the numbers given in Question 1b) and why did you choose those two?* (e.g., I only want the base and height, not the other side length for the parallelogram as it is not used in the formula.) ◊ *Why did you need to know both bases of the trapezoid, as well as the area, to figure out the height in Question 3?* (e.g., Because whatever the other base is, you could adjust the height to get the right area, but the heights would be different.) ◊ *How do you decide which base and height to use when you are calculating the area of a triangle?* (e.g., I used whichever one I want. It does not matter.) *Could you use different bases and heights on a parallelogram too?* (e.g., I did not try, but I think I could have.) ◊ *What did your three trapezoids for Question 7 have in common (besides the area)?* (e.g., I gave them all the same height.) *Did you have to?* (No.) ◊ *Are you sure you could divide any trapezoid into two triangles as you did in your answer to Question 11?* (e.g., Yes, since you would draw a line from one vertex to the opposite one.) *What would the bases and heights of those triangles be?* (e.g., The bases would be the two bases of the trapezoid, one for each triangle, and they would have the same height – the height of the trapezoid.)

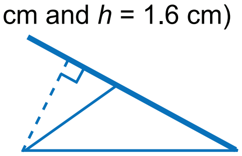
Solutions

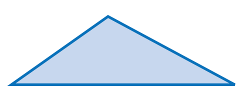
1. **a)** 15 square units **b)** 3 square units
c) 30 square units **d)** 24 square units

2. **a)** 32 square units **b)** 12 square units
c) 20 square units

3. **a)** 6 cm **b)** 12 cm **c)** 5 cm

4. **a)** e.g., 4 cm^2 (b = 5 cm and h = 1.6 cm)
b) new base = 3.6 cm
c) height must be $\frac{8}{3.6}$
d) new height = 2.2 cm

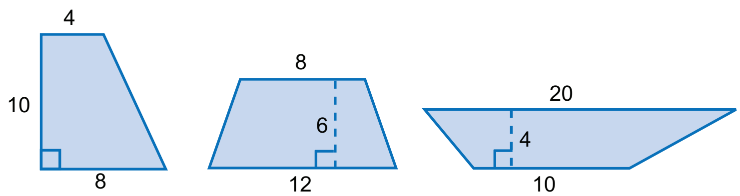




5. The height of the triangle is twice as much, since you take half of the product of the base and height with the triangle. You do not do that with the parallelogram.

6. e.g., The shaded triangle is half as high and half as wide. So in the formula, you would divide by 2 twice (if you started with the white triangle) and that is the same as taking $\frac{1}{4}$.

7. e.g.,

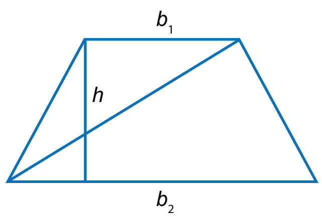


8. 23.4 m^2; e.g., The hexagon is made up of two trapezoids. Each has a base of 3 and 6 and a height of 5.2 ÷ 2.

9. 8.25 cm^2

10. e.g., The sum of the bases must be 20 cm.

11. The area of the top triangle is $\frac{1}{2}b_1h$. The area of the bottom triangle is $\frac{1}{2}b_2h$.
The total area = $\frac{1}{2}b_1h + \frac{1}{2}b_2h$.



Open Question

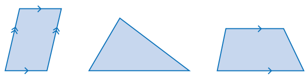
Areas of Parallelograms, Triangles, and Trapezoids

Learning Goal

- recognizing the efficiency of using formulas to simplify area calculations.

Open Question

A parallelogram, a triangle, and a trapezoid (none with a right angle), all have the same area.



- **Step 1:** Choose the area; ________ cm².
  Draw sketches and mark the dimensions (bases, heights) so that your three shapes will have this area. Show why the shapes have the required area.

- **Step 2:** Repeat using the same area, but different dimensions, for your three shapes.

- **Step 3:** Repeat Steps 1 and 2, using two different areas.

6 September 2011 © Marian Small, 2011 2 - D Measurement (IS)

Think Sheet

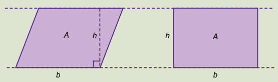
Think Sheet

One way to measure the area of a shape is to place that shape on a grid and count squares.

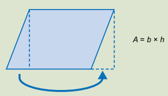
We can also use a formula that allows us to calculate the area using only length measures.

Area of a parallelogram

Consider a rectangle with the same base and the same height as the parallelogram. We know that the area of the rectangle is its length multiplied by its height.



The parallelogram can be rearranged into the rectangle. So knowing a parallelogram's base length and height allows us to calculate its area using that same formula we use to calculate the area of a rectangle.



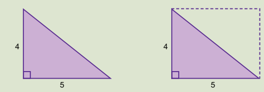
Notice that the height of the parallelogram is shorter than the slanted side length. It is important to use a base and the height, and not two side lengths, in the formula.

If we know the area of a parallelogram and the base or height, we can figure out the other dimension by dividing. For example, a parallelogram with an area of 20 cm^2 and a base of 4 cm has to have a height of 20 ÷ 4 = 5 cm.

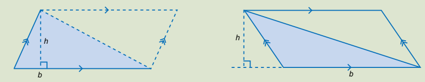
Area of a triangle

To figure out the areas of right triangles we might notice that they are halves of rectangles.

For example, the area of the triangle has to be $\frac{1}{2}$ of 4 × 5:

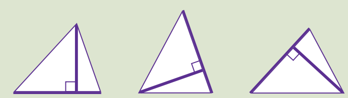


Other triangles are not as easy to visualize in terms of rectangles. These triangles might be viewed as half parallelograms.



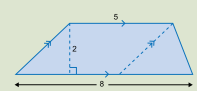
Each triangle is half of a parallelogram with a base that matches the triangle's base and a height that matches the triangle's height. [Sometimes, as with the triangle on the right, the height has to be drawn outside of the shape by extending the base.]

Any side of a triangle can be considered the base of that triangle, but the height that is used in the formula must be the height that is perpendicular to that base. There are three possible base-height combinations for a triangle:



Area of a trapezoid

Every trapezoid can be divided into a parallelogram and a triangle. The area of the trapezoid can be calculated by adding the area of that parallelogram and the area of that triangle. For example:



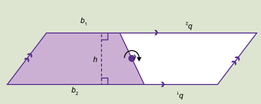
The 8 units in the base can be divided into 5 units for the base of the parallelogram and 3 for the base of the triangle. The heights of the parallelogram and triangle are 2 units.

The area of the parallelogram is 2 × 5 square units.

The area of the triangle is $\frac{1}{2}$ of 3 × 2 square units.

So the area of the trapezoid is 2 × 5 + $\frac{1}{2}$ of 3 × 2 = 13 square units.

We can also think of any trapezoid as half of a parallelogram; turn a copy of the trapezoid around 180° and let the sides of the original and the copy touch, as shown:



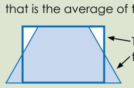
The base of the parallelogram is the sum of the two bases of the trapezoid ($b_1 + b_2$).

The height of the trapezoid is the height of the parallelogram.

But the area of the trapezoid is only half of the area of the parallelogram, so the area of the trapezoid is $\frac{1}{2}(b_1 + b_2)h$.

The area for the trapezoid above would be $\frac{1}{2}(5 + 8) \times 2 = 13$ square units.

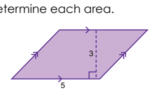
We can also think of the formula as: Take the average of the two bases and multiply by the height. This makes sense if you think of the rectangle with a base that is the average of the two bases and the same height as the parallelogram.



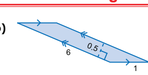
The extra white areas balance the areas of the small triangles on each side that are not inside the rectangle.

1. Determine each area.

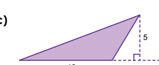
a)



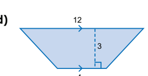
b)



c)

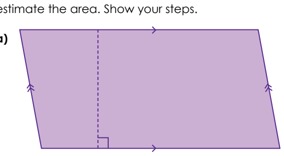


d)

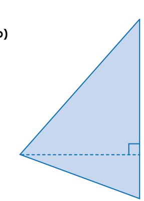


2. Use your ruler to measure the lengths you need to determine each area. Then estimate the area. Show your steps.

a)

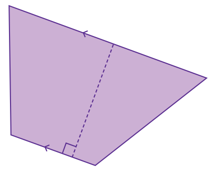


b)



Areas of Parallelograms, Triangles, and Trapezoids **(Continued)**

c)



3. What is the height of each shape?

a) a parallelogram with an area of 30 cm^2 and a base of 5 cm

b) a triangle with an area of 30 cm^2 and a base of 5 cm

c) a trapezoid with an area of 30 cm^2 and bases of 4 cm and 8 cm

4. a) Draw a non-right triangle and calculate its area.

b) Use a different side of the triangle as the base. Measure it.

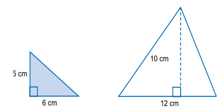
c) Tell what the height that matches that base must be and why.

d) Measure to check.

Areas of Parallelograms, Triangles, and Trapezoids **(Continued)**

5. A parallelogram and a triangle have the same base and the same area. What do you know about their heights? Explain why.

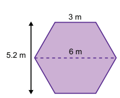
6. How can you tell that the shaded triangle has $\frac{1}{4}$ the area of the white one without calculating each area?



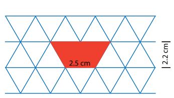
7. Sketch three different trapezoids, each with an area of 60 cm². Mark the base lengths and heights for each.

Areas of Parallelograms, Triangles, and Trapezoids **(Continued)**

8. What is the area of this regular hexagon? How do you know?



9. What is the area of the red pattern block?



10. A trapezoid has an area of 100 cm^2. Its height is 10 cm. What else do you know about the measurements of the trapezoid?

11. Draw any trapezoid. Divide it into two triangles. Create a formula for the area of a trapezoid based on the areas of those two triangles. Show your reasoning.

Areas of Composite Shapes

Learning Goal

- composing and decomposing shapes to simplify area calculation.

Open Question

Materials
- ruler
- calculator (optional)
- Formula Sheet
- calculators (optional)

Questions to Ask Before Using the Open Question

◊ *Draw a trapezoid. What familiar shapes make up that trapezoid?* (e.g., a parallelogram and a triangle.)
◊ *What other shape can you make by putting together parallelograms and triangles?* (e.g., I can make a 7-sided shape by putting two parallelograms side-by-side and putting a triangle on part of them in the middle.) *How would you get the area of that new shape?* (e.g., I would add the areas of the parallelograms and the triangle.)
◊ *Suppose you rearranged those three shapes into a different shape. How would the area of the shape change?* (It would not change.)
◊ *If a parallelogram were inside a rectangle and you wanted to find the area of the part of the rectangle that was not part of the parallelogram? What would you do?* (e.g., I would figure out the area of the rectangle and subtract the area of the parallelogram from it.)

Using the Open Question

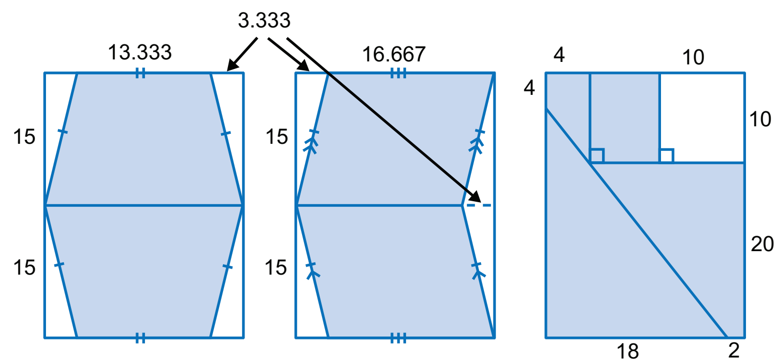
Encourage students to use triangles, parallelograms, and trapezoids, in their drawings. They may also choose to use rectangles.

By viewing or listening to student responses, note if they:
- can put together areas of known shapes to determine the area of a composite shape;
- can subtract one area from another to determine the area of a third shape;
- can correctly apply the formulas for areas of parallelograms, triangles, and trapezoids;
- recognize when determining one measure of a shape can indirectly provide another measure of that shape or an adjacent shape.

Consolidating and Reflecting on the Open Question

◊ *How did you start solving the problem?* (e.g., I drew two trapezoids, one on top of the other, that almost filled the whole rectangle. I had to make sure that I left the right amount blank.)
◊ *How did you do that?* (e.g., I figured out that the area of the rectangle was 20×30 cm². I knew that, since I wanted a total area of 500 cm², there would have to be 100 cm² blank. When I saw that I had four identical triangles left, I figured out what the dimensions had to be so that each one had an area of 25 cm².)
◊ *Why did it make sense that the bases of those left-over triangles were so small?* (e.g., The area was not that big and the height was big.)
◊ *Was your thinking any different for your other designs?* (e.g., It was a little different for the last one, since I did not have two identical shapes.)
◊ *If the shape had been outlined without the lines inside, would it have been easy to see how to divide up the shapes into ones you knew?* (e.g., I think it would have been easier for the first two designs. I am not sure I would have seen a rectangle in the last design.)
◊ *Once you knew that the small trapezoid in your last shape had a height of 4, why did you automatically know the width of the rectangle next to it?* (e.g., The three widths at the top had to add to 20.)

Solutions



e.g.

Shape 1

The area of the 20 cm × 30 cm rectangle is 600 cm^2, so I realized that if I wanted a shape with an area of 500 cm^2, there would be 100 cm^2 to subtract.

In the first picture, I made up a hexagon out of two trapezoids. I saw that there were four triangles left over that were identical, so each needed to have an area of 25 cm^2.

I knew the heights were half the height of the rectangle, so the heights were 15 cm and I marked one of them. I had to figure out how to make the base multiplied by 15 and divided by 2 work out to 25. I used an equation: $\frac{15b}{2} = 25$, and solved it; $b = 3.333$. Since all four triangles were the same, I only needed to mark the base and height in one of them.

Since there were two identical trapezoids, the area of each one had to be 250 cm^2. I confirmed this by realizing that the height of each was 15 cm and the bases were 20 cm and 13.33 cm. I used the formula: $\frac{15 \times (20 + 13.333)}{2} = 250$.

Shape 2

In the second picture, I made up a hexagon using two parallelograms. Again, I had four identical triangles left over by dividing up the triangle on the right into two small right triangles. I was able to do the same thing as in the first picture.

Since there were two identical parallelograms, the area of each had to be 250 cm^2. The base was 16.6667 cm and the height was 15 cm. I confirmed the area using the formula: 15 × 16.6667 = 250.

Shape 3

In the third picture, I used one triangle, two trapezoids, and a rectangle. I left a 10 × 10 white square, so I knew there was 500 cm^2 that was shaded.

I knew that the triangle had area $\frac{1}{2} \times 26 \times 18$ cm^2 = 234 cm^2.

I knew that the bottom trapezoid had an area of 20 × $\frac{1}{2}$ of (2 + 16) cm^2 = 180 cm^2.

I knew that the top trapezoid had an area of 4 × $\frac{1}{2}$ of (4 + 10) cm^2 = 28 cm^2.

I knew that the rectangle had an area of 10 × 6 cm^2 = 60 cm^2.

The total area is 502 cm^2. I must have overlapped the shapes just a bit where they met on the diagonal, but I thought this was close enough.

Think Sheet

Materials
- ruler
- calculator (optional)
- Formula Sheet

Questions to Ask Before Assigning the Think Sheet

◊ *Draw a trapezoid. What familiar shapes make up that trapezoid?* (e.g., a parallelogram and a triangle)
◊ *What other shape can you make by putting together parallelograms and triangles?* (e.g., I can make a 7-sided shape by putting two parallelograms side-by-side and putting a triangle on top of them in the middle.) *How would you get the area of that new shape?* (e.g., I would add the areas of the parallelograms and triangle.)
◊ *Suppose you rearranged those three shapes into a different shape. How would the area of the shape change?* (It would not change.)
◊ *If a parallelogram were inside a rectangle and you wanted to find the area of the part of the rectangle that was not part of the parallelogram? What would you do?* (e.g., I would figure out the area of the rectangle and I subtract the area of the parallelogram from it.)

Using the Think Sheet

Read through the introductory box with the students and make sure they understand the material explained in the instructional box.

Assign the tasks.

By viewing or listening to student responses, note if they:
- can decompose a composite shape into simpler components;
- can determine which linear measures are needed to determine the area of the components of the composite shape;
- can calculate the area of shapes made up of parallelograms, triangles, and trapezoids;
- recognize that knowing some measurements of a shape can determine other measures of that shape or an attached shape.

Consolidating and Reflecting: Questions to Ask After Using the Think Sheet

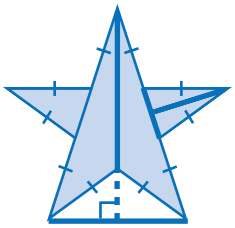
◊ *Was there more than one way to divide up the shapes in Question 1?* (Yes, e.g., the first shape could be a rectangle and a triangle, but it could also be a trapezoid and two triangles.)
◊ *Why did you only need some measurements of the star in Question 3, but not all of them, to determine the area?* (e.g., The star is symmetric so if you know about one side, you also know about the other side.)
◊ *Could you have chosen different measurements to use?* (e.g., Yes, I could have chosen different bases and heights for the triangles I made in question 3, although I think the ones I chose were good ones.)
◊ *When did you use subtraction to help you figure out an unknown area?* (e.g., If there was a white shape completely inside a coloured one.)
◊ *What ideas about the area formulas did you use to help you answer the last question?* (e.g., I know that if you have the same base and a half-height, you have a half-area, so I used that for part a.)

Solutions

1. **a)** a triangle on top of a rectangle
b) a rectangle subtracted from a rectangle
c) a square on the inside and four identical triangles
d) a parallelogram subtracted from a parallelogram
e) 1 square at the top left, 1 square at the bottom right and a right triangle on top right and another on bottom left

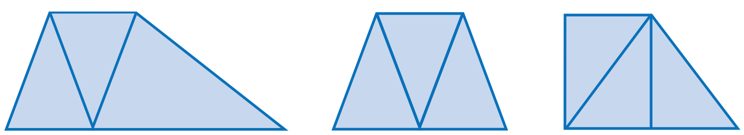
2. **a)** 47.5 square units
b) 22 m^2
c) 27 m^2
d) 12 square units
e) 48 cm^2

3. e.g.,
a) There is a large triangle. You take off the small triangle at the bottom. You add in the two triangles on the side.
b)



c) Area = 10 cm^2

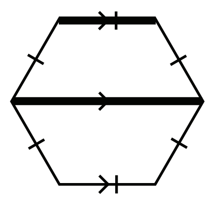
4. **a)** I used an scalene trapazoid, an isosceles trapazoid, and a right trapazoid. I think that's all the kinds.
e.g.,



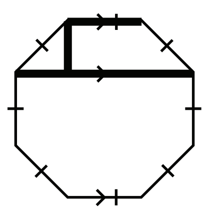
b) e.g., You would need to know the height and both bases for all of them.
For the first two pictures, if you know the bottom base, it does not matter where it is split since both parts are multiplied by the same height and then added.

5. **a)** Yes, e.g., If you know the side at the left, you know the height and can use the formula for the area of a trapezoid.
b) No, since you would not know the height.

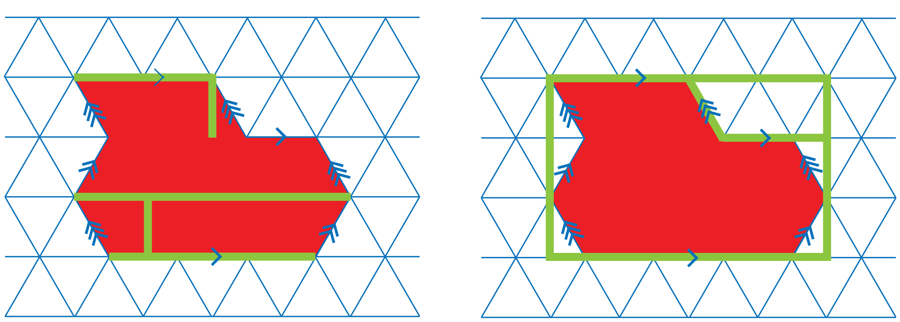
6. **a)** e.g., You could divide it into two trapezoids. You would still need to know the side length of the hexagon and its width. You could figure out the height of each trapezoid using the Pythagorean theorem.



b) e.g., You could divide it into two trapezoids and a rectangle. You would need to know the side length of the octagon and its width. You could figure out the height using the Pythagorean theorem.



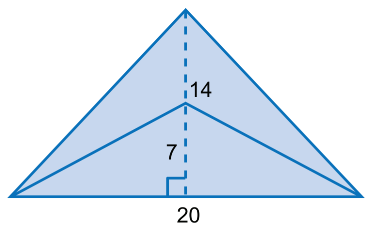
7.



The first time, I have two trapezoids and a parallelogram. I need the base and height of the parallelogram (the top two lines I marked). The two trapezoids are the same so I need the two bases and height of one of them.

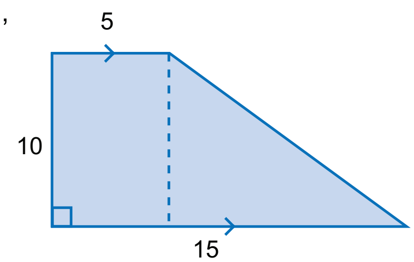
For the second picture, I need the base and height of the large rectangle from which I am going to subtract. I need the two base lenghts of the white trapaziod at the top right. If I also know the base and height of the bottom left white triangle, I know the one on the right, since they are the same size. Knowing the height of the bottom left triangle gives me the height on the top left, so that's all I need.

8. **a)** e.g.,



These two shapes have equal area since the bottom shape is a triangle with base 20 and height 7; its area is 70 square units. The entire shape is a triangle with base 20 and height 14; its area is 140 square units. So the piece at the top must have an area of 70 square units.

b) e.g.,



The shape on the left is a rectangle with area 5 × 10 = 50 square units.

The triangle on the right has a base of 10 and height of 10, so it is area is also 50 square units.

Open Question

Areas of Composite Shapes

Learning Goal

- composing and decomposing shapes to simplify area calculation.

Open Question

A composite shape is made up of a combination of some of these shapes: rectangles, triangles, parallelograms, and trapezoids.

Create composite shapes using these conditions:

- There is always at least one parallelogram or trapezoid as part of the shape.
- The shape fits inside a rectangle that is 30 cm × 20 cm.
- The total area of the shape is 500 cm².

- Draw at least three (or more) possible shapes. Use a marker or coloured pencil to mark on your drawings those measurements of the rectangles, triangles, parallelograms, and trapezoids you would need to know to be sure that the area is 500 cm² without using a grid.

- Tell what the area of each component must be, using formulas for the areas of those particular shapes.

Think Sheet

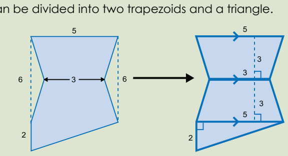
Areas of Composite Shapes (Continued)

Think Sheet

Composite shapes are shapes that are made by putting together other shapes.

If a composite shape is made up of rectangles, triangles, trapezoids, and/or parallelograms, we can use what we know about the areas of the simpler shapes to figure out the area of the composite shape.

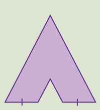
For example, it is challenging to figure out the area of the shape on the left unless we notice it can be divided into two trapezoids and a triangle.

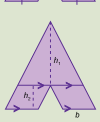


The area of each trapezoid is $\frac{1}{2}(3 + 5) \times 3$ square units. The height of 6 is divided — half for each trapezoid. The area of the triangle is $\frac{1}{2}$ of $2 \times 5 = 5$ square units. So the total area is 29 square units.

It is important that the dimensions we need for the simpler shapes are given or that we have enough information to figure them out. Notice that we had to figure out that the height of each trapezoid was 3 from what was given.

The area of this shape can be calculated if only a few lengths are known:





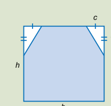
We can divide the shape into a triangle and two parallelograms. If we know the base of one of the parallelograms, *b*, and the two indicated heights, we can calculate the area. That is because the base of the triangle is twice the base of the parallelogram. So if we know the base of the triangle, we can calculate the base of the parallelogram.

$A = bh_2 + bh_2 + \frac{1}{2}h_1 \times 2b$

$A = 2bh_2 + bh_1$

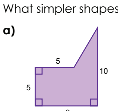
Areas of Composite Shapes (Continued)

Sometimes we take away the area of a known shape to calculate the area of a more complicated shape. For example, the area of the shaded shape below is $bh - 2 \times \frac{1}{2}cd$.

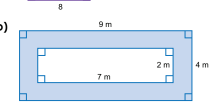


1. What simpler shapes could you use to describe the shaded shape?

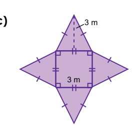
a)



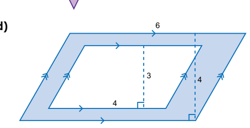
b)



c)

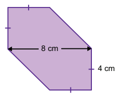


d)



Areas of Composite Shapes (Continued)

e)



2. Calculate the areas of the shapes in Question 1.

a)

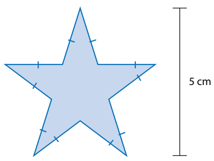
b)

c)

d)

e)

3. a) How would you divide up this star into simpler shapes to make it easier to calculate its area?



b) Use your ruler to measure. Mark the measurements that you would need to calculate the areas of the pieces of the star.

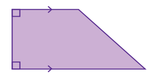
c) Estimate the area of the star?

Areas of Composite Shapes (Continued)

4. a) Prove that every trapezoid can be divided into three triangles.

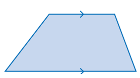
b) What dimensions would you need to know to calculate the area of the trapezoid based on how you divided it up in part a)? Explain.

5. a) If you know all of the side lengths of a trapezoid with two right angles, but no other measurements, can you figure out the area without doing any more measurements? Explain.

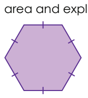


Areas of Composite Shapes

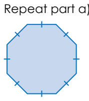
b) If you know all of the side lengths of a non-right trapezoid but no other measurements, can you figure out the area without doing any more measurements? Explain.



6. a) How could you divide a regular hexagon into simpler shapes to calculate its area? Mark the fewest dimensions you would need to know to figure out the area and explain.

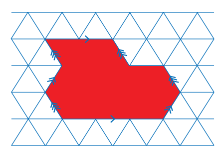


b) Repeat part a) with a regular octagon.



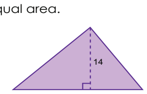
Areas of Composite Shapes (Continued)

7. Show two different ways to measure lengths so you can calculate the total area. Explain your thinking.

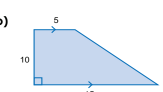


8. Divide each shape into two shapes of equal area. Prove that the shapes have equal area.

a)



b)



Circumferences and Areas of Circles

Learning Goal

- recognizing that knowing one measure of a circle provides information about all other measures.

Open Question

Materials
- rulers
- calculators
- grid paper
- string, scissors
- compass (optional)
- Four Circles template (optional)

Note: Students should not have access to the Formula Sheet template for this activity since they are developing the formulas.

Questions to Ask Before Using the Open Question

◊ *Draw a square. How does the perimeter of this square compare to the side length?* (e.g., It is four times as much.) *Why is that?* (e.g., There are four equal sides compared to just one.)
◊ *Draw an equilateral triangle. How does the perimeter of the triangle compare to the side length?* (e.g., It is three times as much.) *Why?* (e.g., There are three equal sides compared to just one.)
◊ *Draw a circle. How might you measure the distance around the circle?* (e.g., I could cut a piece of string that I wrap around it and then straighten out the string and measure it with a ruler.)
◊ *If the circle is on grid paper, how might you measure the area?* (e.g., I would outline a rectangle in the middle to quickly count that part of the area and then I would count the other squares and part squares. I know I would have to estimate.)

Using the Open Question

Students trace around round objects or can use a compass to construct their four circles. Alternately, provide the Four Circles template.

By viewing or listening to student responses, note if they:
- observe the constant ratio (of about 3 circles) relating the diameter and circumference of a circle;
- observe the constant ratio (of about 3 circles) relating the area of a circle to the square of its radius;
- can visually represent a comparison between the circumference of a circle and its diameter;
- can visually represent a comparison between the area of a circle and the area of a square with the radius as side length.

Consolidating and Reflecting on the Open Question

◊ *What did you notice about the circumference of a circle with a bigger radius?* (e.g., It was bigger.)
◊ *Was the same true about the areas? (Yes)*
◊ *What was the same about all of the circumferences (besides that they were round)?* (e.g., They were all just a little more than three times the diameter.)

Introduce the number pi, π, to students as the number that describes that ratio.
◊ *Where else did that number π come up? (e.g., It came up when I was comparing the area of the square to the area of the circle.)*
◊ *How did your drawing help show the relationships you found?* (e.g., My picture shows that I would need more than three pieces of string the length of the diameter to go around the circle, so that shows π. My other picture shows that a square on the radius looks like it fits into a bit more than $\frac{1}{4}$ of the circle.)
[If students have a drawing other than the one in the solutions, introduce the drawing below as well and ask: How does this picture help you see that the circumference of a circle is less than 4 times its diameter, but not a lot less? How does it help you see that the area of a circle is less than 4 times the area of a square drawn on the radius, but not a lot less? (See Solutions for example of a response.)

Solutions

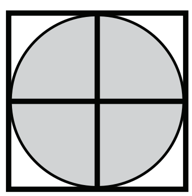
e.g., I drew circles with diameters of 5 cm, 10 cm, 15 cm, and 20 cm.

	Circle 1	Circle 2	Circle 3	Circle 4
Diameter	5 cm	10 cm	15 cm	20 cm
Distance around	15.5 cm	31 cm	47 cm	62 cm
Distance around compared to diameter	3.1 times as big	3.1 times as big	3.1 times as big	3.1 times as big
Area estimate	75 cm^2	310 cm^2	700 cm^2	1250 cm^2
Square area	25 cm^2	100 cm^2	225 cm^2	400 cm^2

The circle area is about three times as much as the square area.

If I draw this picture, it looks like the area of the circle is not as much as the area of 4 squares, but almost as much.

Using the same picture, I see that the square has perimeter 4d, where d is the diameter of the circle. But I can see that the circumference of the circle does not "go out" as much at the corners, so it has to be less than 4d, but not a lot less.



Think Sheet

Materials
- rulers
- calculators
- string, scissors
- Divided Circle template
- Formula Sheet

Questions to Ask Before Assigning the Think Sheet

◊ *Draw a square. How does the perimeter of the square compare to the side length?* (e.g., It is four times as much.) *Why is that?* (e.g., There are four equal sides compared to just one.)
◊ *Draw an equilateral triangle. How does the perimeter of the triangle compare to the side length?* (e.g., It is three times as much.) *Why?* (e.g., There are three equal sides compared to just one.)
◊ *Show the students a circle. How do you think this perimeter (or circumference) will compare to the diameter of the circle?* (e.g., It looks like it might be three or four times as much.)

Using the Think Sheet

Read through the introductory box with the students and make sure they understand the material explained in the instructional box.

Assist students to cut a string the length of the circumference of a circle and see that a diameter length fits into it three times and a bit. Also assist students in cutting out a circle into eight equal sectors, rearranging them as is suggested in the section on area. You can use the Divided Circle template.

Assign the tasks.

By viewing or listening to student responses, note if they:
- can apply the formulas for the circumference and area of circles;
- can use the circumference or area formula backwards to determine a radius or diameter given the circumference or area;
- recognize that if one linear measure of a circle doubles, so do other linear measures, but that this is not true of the area;
- can determine the circumference or area of a composite shape involving circles or parts of circles;
- have some sense of how quickly circle areas grow;
- recognize the interconnectedness of the various formulas involving measures of a circle.

Consolidating and Reflecting: Questions to Ask After Using the Think Sheet

◊ *How did you figure out the radii or diameters you needed to use in Question 2?* (e.g., For part a), I could see the diameter was the side of the square and it was marked as 10 cm. For part b), I had to figure out that the part of the top side of the square that was missing was 10 cm, so that was the radius of the quarter circle. For part c), I saw that the diameter was 50 metres, since all the sides of the square were equal.)
◊ *How do you get the radius of a circle if you know its circumference?* (e.g., Since the circumference is π times the diameter, I would divide by π to get the diameter and then I would divide by 2 to get the radius.) *What if you know the area?* (e.g., I would divide by π to get the square of the radius and then I would take its square root.)
◊ *How did you solve Question 8?* (e.g., I used an equation. I knew that if I called the black circle radius r, that $\pi r^2 = \frac{400\pi}{5}$.)
◊ *How did you figure out the diameters or radii in Question 10?* (e.g., I could see that 6 touching circles made 18 cm, so each circle had to have a diameter of 3 cm. That also told me that the width of the rectangle was 6 cm since there were two touching circles making up the width.)
◊ *How are all of the measurement formulas for a circle related?* (e.g., They all involve π and they all involve either r or d.)

Solutions

Note: All solutions are given both in terms of π as well as using an estimate of 3.14 for π.

1. **a)** 8π cm (or 25.12 cm) **b)** 10π cm (or 31.4 cm)

2. **a)** 20π cm (or 62.8 cm) **b)** 60 + 5π cm (or 75.7 cm)
c) 100 + 100π m (or 414 m)

3. The black ribbon length is 10π cm (or 31.4 cm) long. The green ribbon length is also 10π cm (or 31.4 cm) long. This is because the black is half of a circle with circumference 20π cm (or 62.8 cm) long and the green is the full circumference of a circle with diameter 10 cm and circumference 10π (or 31.4) cm

4. radius = $\frac{20}{2\pi}$ cm (or 3.18 cm), since the circumference is twice the radius (diameter) × π

5. It is also double; e.g., In the formula if you substitute 2*d* for *d* in $C = \pi d$, you get $2\pi d$, which is twice the circumference of the original circle.

6. **a)** $100 + 2\pi \times 25 = 100 + 50\pi$ cm² (or 257 cm²)
b) $400 - \frac{1}{4}(100\pi) = 400 - 25\pi$ cm² (or 321.5 cm²)
c) 2500 + 625π cm² (or 4462.5 cm²)

7. **a)** $r = 3$ m, so $\frac{1}{4} \times 9\pi = \frac{9\pi}{4}$ m² (or 7.1 m²)
b) Take away the area of the circle with radius 5 from the area of the square: 100 – 25π m² (or 21.5 m²).
c) Add the semi-circle area to the area of a trapezoid: $\frac{9\pi}{2} + 30$ cm² (or 44.1 cm²).
d) Take away the white semi-circle area away from a semi-circle area as wide as the grey section. The radius of the larger semi-circle is 7 metres and the radius of the smaller one is 5 m. $49\frac{\pi}{2} - 25\frac{\pi}{2} = 12\pi$ cm² (or 37.68 cm²)

8. e.g., almost 9 cm

9. $\sqrt{\frac{10}{\pi}}$ cm (or 1.8 cm)

10. $18 \times 6 - 12 \times \pi \times 1.5^2 = 108 - 27\pi$ cm = 23.2 cm²

11. $\sqrt{100\pi}$ cm (or 17.7 cm)

12. The bigger radius is 3 times the smaller one.

13. e.g., All of the formulas depend only on one value, the value of *r*.

If you know *r*, you know that $d = 2r$, so $C = 2\pi r$, and $A = \pi r^2$.

If you know *d*, you know that $r = \frac{d}{2}$ and can substitute $\frac{d}{2}$ for *r* in the formulas for area and circumference above.

If you know *C*, you can replace *r* in the formulas above with $\frac{C}{2\pi}$.

If you know *A*, you can replace *r* in the formulas for *d* and *C* with $\sqrt{\frac{A}{\pi}}$.

Open Question

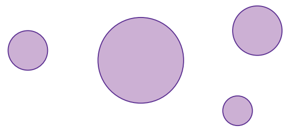
Circumferences and Areas of Circles

Learning Goal

- recognizing that knowing one measure of a circle provides information about all other measures.

Open Question

Draw at least four circles of different sizes.



- For each circle, measure the distance around it. Compare that measurement to the diameter of that circle.
- For each circle, estimate the area of the circle using grid paper. Compare it to the area of a square whose side length is the radius of the circle.
- Describe what you observed. Draw pictures to show why what you observed makes sense.

Think Sheet

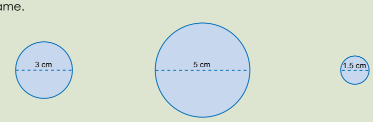
Circumferences and Areas of Circles (Continued)

Think Sheet

Measuring the perimeter of circles is more challenging than measuring the distance around polygons because we cannot use a ruler. Even a tape measure is awkward. It is for this reason that formulas for the circumference (perimeter) of a circle and its area are useful.

Circumference

The circumference of a circle is the distance around it. A string was used to measure the circumferences of these circles. We might notice that for each one, the ratio of the circumference to the diameter (the distance across the circle) is the same.



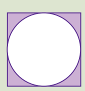
Estimated Circumference = 9.4 cm	Estimated Circumference = 15.7 cm	Estimated Circumference = 4.7 cm
Ratios: $\frac{9.4}{3} = 3.14$	$\frac{15.7}{5} = 3.14$	$\frac{4.7}{1.5} = 3.14$

The exact ratio is a number called π that is pronounced ***pi***.

For any diameter, we multiply by π to figure out the circumference.

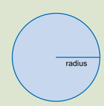
This diagram of a circle inside a square might help us see why this ratio makes sense.

Notice that the perimeter of the square with a side length the diameter of the circle is 4 times the diameter. It seems reasonable the circle's circumference is less than 4 times that diameter.



Circumferences and Areas of Circles (Continued)

The radius of the circle is the distance from the centre to the edge. It is always half of the diameter.

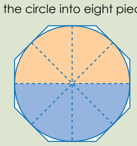


So $C = \pi d$ or $C = 2r \times \pi$ ($C = 2\pi r$)

Area

Knowing the radius or diameter of a circle allows us to figure out its area.

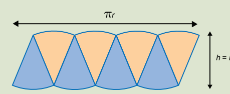
It helps to draw the circle inside a polygon. For example, we could draw the circle inside an octagon and cut the circle into eight pieces.



If we rearrange the eight pieces, we can create a shape that is "almost" a parallelogram. If a polygon with more sides were used, the curve on the bases would be less.

The height of the parallelogram is the radius of the circle.

Each base length of the parallelogram is half the circumference since the two bases make up the whole circumference.

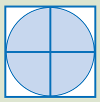


$C = 2\pi r$, so $\frac{1}{2}C = \pi r$.

Circumferences and Areas of Circles (Continued)

The area is the product of the base and height, so $A = \pi r \times r = \pi r^2$.

This picture might help us see why it makes sense that the area of a circle is about three times the area of a square on its radius. The circle's area is less than the area of four squares with a side length equal to the radius, since there are white sections not covered by the circle. But it is not a lot less.

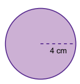


If we know the area of the circle, we can calculate the radius.

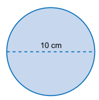
$r^2 = \frac{A}{\pi}$ so $r = \sqrt{\frac{A}{\pi}}$.

1. Calculate the circumference of each circle.

a)

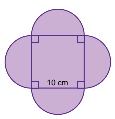


b)

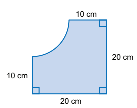


2. Determine the perimeter of each shape.

a)

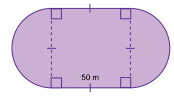


b)

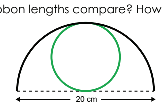


Circumferences and Areas of Circles (Continued)

c)



3. Black ribbon is used for the semi-circle of this design. Green ribbon is used for the circle. How do the two ribbon lengths compare? How do you know?



4. A circle has a circumference of 20 cm. What is its radius? How do you know?

5. If the diameter of one circle is double the diameter of another, how do their circumferences compare? Explain.

6. Determine the area of each shape in Question 2.

a)

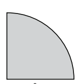
b)

c)

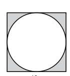
Circumferences and Areas of Circles (Continued)

7. Calculate the area of the shaded space. Show your thinking.

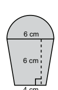
a)



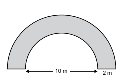
b)



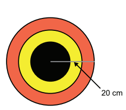
c)



d)



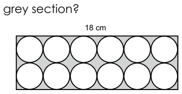
8. The black section of the dart board is $\frac{1}{5}$ of the total area. What is the radius of the black section?



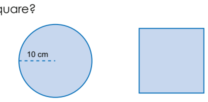
9. A circle has an area of 10 cm^2. What is its radius?

Circumferences and Areas of Circles (Continued)

10. What is the area of the grey section?



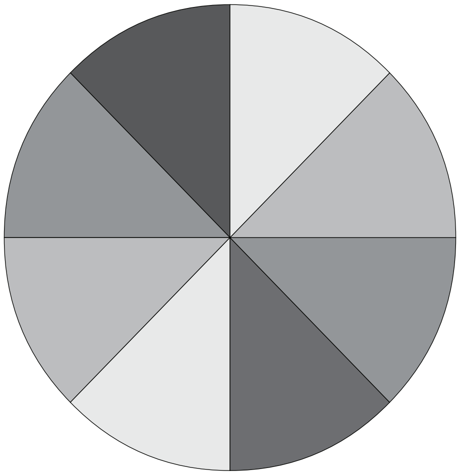
11. A square has almost the same area as a circle with radius 10 cm. What is the side length of the square?



12. Circle A has an area 9 times the area of Circle B. What do you know about their radii (the plural of radius)? Explain why your answer makes sense.

13. Why do you only need one of the radius, diameter, circumference, or area of a circle to figure out all of the others?

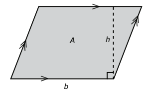
Divided Circle



Formula Sheet

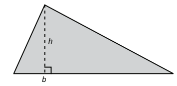
Area of a parallelogram

A = *bh*



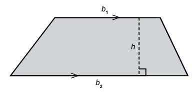
Area of a triangle

$A = \frac{1}{2}bh$ or $A = bh \div 2$



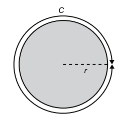
Area of a trapezoid

$A = \frac{1}{2}(b_1 + b_2)h$ or $A = (b_1 + b_2)h \div 2$



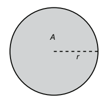
Circumference of a circle

$C = 2\pi r$ or $C = \pi d$

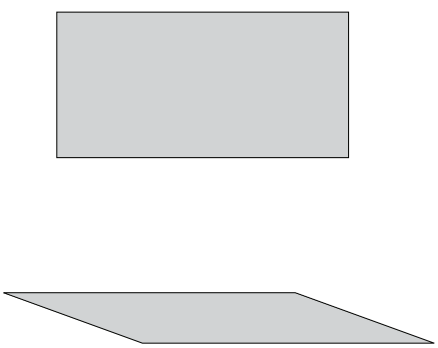


Area of a circle

$A = \pi r^2$



Rectangle and Parallelogram



Four Circles