GAP CLOSING

Proportional Reasoning

Intermediate / Senior
Student Book

Topic 4

Proportional Reasoning

This entire module, including the Diagnostic plus all Intervention Materials, can be downloaded at http://www.edugains.ca/resources/LearningMaterials/GapClosing/Grade9/4-PropReasoning_SB_IS.pdf. The Facilitator's Guide for this module can be downloaded at http://www.edugains.ca/resources/LearningMaterials/GapClosing/Grade9/4-PropReasoning_FG_IS.pdf.

Diagnostic

1. There are 8 boys and 3 girls on the Tech Team.

   a) Write the ratio of he number of girls to number of boys in the form □ : □.

   b) Write the ratio of the number of boys to the number on the whole team.

   c) Another Tech Team of 11 students has a higher ratio of number of girls to number of boys. What could the ratio be?

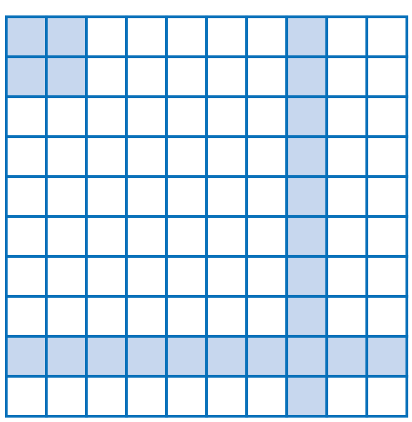
2. The ratio of Vada's height to Melissa's height is 5 : 4.

   a) Who is taller?

   b) Is she twice as tall? How do you know?

3. Valene's running rate is 0.18 km/min. Explain what that means.

4. **a)** What percent of the grid is shaded?



   b) What percent is not shaded?

5. Indicate whether each statement does or does not make sense by circling your choice.

a) 8% of something is a lot of it.

MAKES SENSE DOES NOT MAKE SENSE

b) 80% of something is a lot more than half of it.

MAKES SENSE DOES NOT MAKE SENSE

c) 35% of the people in a high school building on a school day are adults, not students.

MAKES SENSE DOES NOT MAKE SENSE

6. Explain your answer to Question 5c.

7. Complete the missing amounts so that the ratios are equivalent.

a) 2 : 7 = ☐ : 14 **b)** 5 : 10 = ☐ : 8 **c)** 12 : ☐ = 3 : 5

8. Suppose your heart beats 144 times in 2 minutes. How many times would you expect it to beat in 5 minutes?

9. What fraction is equivalent to each percent?

a) 40% **b)** 112% **c)** 3.5%

10. Three bars of soap cost $2.61. At this rate, how much would each number of bars below cost?

a) 6 bars

b) 8 bars

11. A car goes 78 km in 45 minutes. At that speed, how far would it go in an hour?

12. A 2.6 L container of juice costs $3.00. How much are you paying for 1 L?

13. Suppose the ratio of the number of boys to the number of girls in a class is 7 : 3. What percent of the class is girls?

14. A T-shirt is priced at $12.99. The store is offering a discount of 30%. How much will the shirt cost (before taxes)?

15. Tell if each statement is TRUE or FALSE by circling the correct word.

a)	40% of 120 is about 30.	TRUE	FALSE
b)	20% of 83 is about 16.	TRUE	FALSE
c)	11% of 198 is about 20.	TRUE	FALSE

16. Explain your answer to Question 15a.

17. Lea spent $25 of the money she saved. She still has 60% of her money left. How much does she have left?

Describing and Representing Ratios, Rates, and Percents

Learning Goal

- representing comparisons based on multiplying as either ratios, rates and percents

Open Question

Ratios, rates, and percents all describe comparisons.

For example:

Ratio – A recipe uses 3 parts flour for every 1 part sugar.

Rate – A painter uses one can of paint to cover 2 walls in 5 hours.

Percent – 52¢ is 52% of 100.

- Search the Internet and find three or four examples for each type of comparison (ratio, rate, and percent) that are related to one of these topics:
  - environmental issues
  - sports
  - the arts
- Each time, indicate what two things are being compared. Write down the url.

Think Sheet

These expressions describe comparisons using a ratio, a rate and a percent:

- Three girls for every four boys is a **ratio** that compares the proportion of girls to boys.

  We can write that ratio as 3 : 4.
  3 and 4 are called **terms** in the ratio: 3 is the first term and 4 is the second term.

  For example, if a class has 12 girls and 16 boys, you could arrange them to show that there are 3 girls for every 4 boys.

  G G G B B B B [3 girls for 4 boys]
  G G G B B B B [3 girls for 4 boys]
  G G G B B B B [3 girls for 4 boys]
  G G G B B B B [3 girls for 4 boys]
  12 girls 16 boys

  Notice that the ratio of girls to all the students is not 3 : 4; it is 3 : 7 since there are 7 students in total for every 3 girls. We could also say $\frac{3}{7}$ of the class is girls.

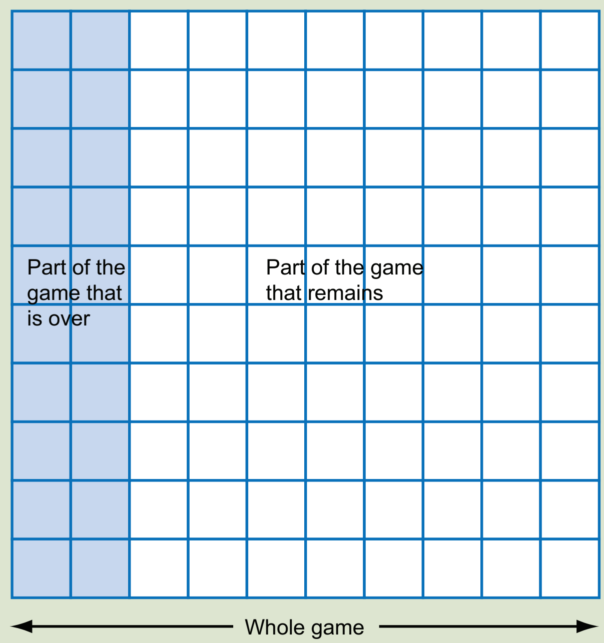
- 3 boxes for $4 is a **rate** that compares an amount of goods to a dollar amount.

  For example, if 3 boxes of one brand of a product costs $4 and 3 boxes of another brand of that product costs $5, we could choose the best buy by comparing the prices of 1 box of each brand or by comparing how much of each brand $1 buys.

  A rate compares two things measured in different units. This time the units were boxes and money. Sometimes rates describe speed (kilometres per hour or metres per second) or a map scale (1 centimetre on the map for every 12 kilometres of real distance.)

- Saying that 20% of the game is over uses a **percent** to compare a part of the game to the whole game. A percent is a ratio where the second term is 100.

  We could also write 20% as the ratio 20 : 100.
  Since percents are ratios out of 100, a 100-grid is a good way to represent a percent. The model shows 20% (20 squares out of 100).



  To compare the part of the game that is over to the part that remains, you could use the ratio 20 : 80 and use the same picture.

1. Draw a picture to show each ratio.

a) 2 circles : 3 squares

b) 4 circles : 5 shapes

2. Which circle is more red? Explain why.

A: 2 parts red out of 6 equal parts.

B: 2 parts red out of 7 equal parts

C: 3 parts red out of 8 equal parts

3. You can mix 1 cup of water with different numbers of cups of orange juice to get different tastes. Which of these ratios of water to orange juice will taste the most watery? Explain why.

1 : 3 1 : 4 1 : 2.5 1 : 3.5

4. Yasir built a scale model of a bird. He decided to use a ratio of:

3 : 2
model lengths : real lengths

a) Was the model bigger or smaller than the real bird? Explain.

b) If a claw on the bird was really 2 centimetres, how long was it on the model?

c) What does the ratio 2 : 3 tell in this situation?

5. The ratio of the length to the width of a rectangle is 12 : 4.

a) Is the rectangle almost square or not? Explain.

b) What is the ratio of the length to the perimeter?

c) Why might the length and width be either 12 centimetres and 4 centimetres or 12 metres and 4 metres, but not 12 metres and 4 centimetres?

6. One measure of fitness is based on comparing your body fat mass to your total mass. A low ratio suggests that you are more fit. Which ratio of body fat mass to total mass is better: 3 : 10 or 3.4 : 10? Explain.

7. Why does your pulse describe a rate?

8. The word ***per*** is often used to describe rates. For example, you might talk about kilometres *per* hour. It can also be shown as a / (e.g., km/hr). List at least three other rates you might describe using the word *per*.

9. What percent might you be representing on a 100-grid if you shade:

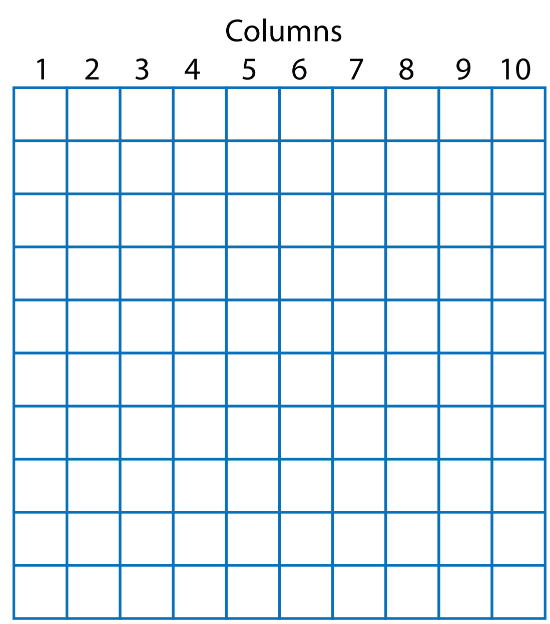
a) every other square?

b) columns 1, 2, 4, 5, 7, and 8?

c) every 4^{th} square?

d) most, but not all, of the squares?

e) just a few squares here and there?



10. On a particular day, 5% of all of the people in a school building are adults. The rest are students.

a) What percent are not adults?

b) What is the ratio of students to adults?

11. Which of these statements make sense? Explain your reasoning for each one.

a) You exercise vigorously 30% of the day.

b) If you flip a coin, it will land on heads 50% of the time.

c) A penny is worth 1% of a loonie.

d) If you buy a sweater and save 5%, you must have saved less than your friend who bought a sweater at 10% off.

Equivalent Forms of Rates, Ratios, and Percents

Learning Goal

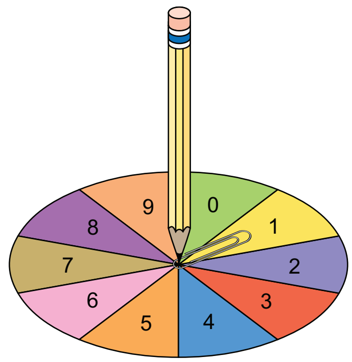
- representing comparisons based on multiplying in a variety of different ways

Open Question

Any fraction can be written in equivalent forms.

For example, $\frac{2}{3} = \frac{6}{9}$, $\frac{9}{15} = \frac{3}{5}$, and $\frac{4}{100} = 0.04$.

Ratios, rates, and percents can also be written in equivalent forms.



- Spin the spinner 9 times to fill in the digits.

Ratio	Rate	Percent
□□ : □□	□□ km/□ h	□□ %

- Show that each can be written in an equivalent form that somewhere includes the number 10.

- Repeat twice more.

Ratio	Rate	Percent
□□ : □□	□□ km/□ h	□□ %

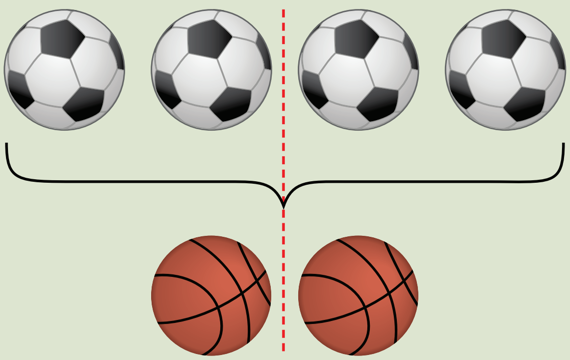
Ratio	Rate	Percent
□□ : □□	□□ km/□ h	□□ %

Think Sheet

There are many ways to describe the same ratio or rate. The **equivalent forms** can be other ratios or rates, or they can be fractions or percents.

Ratio

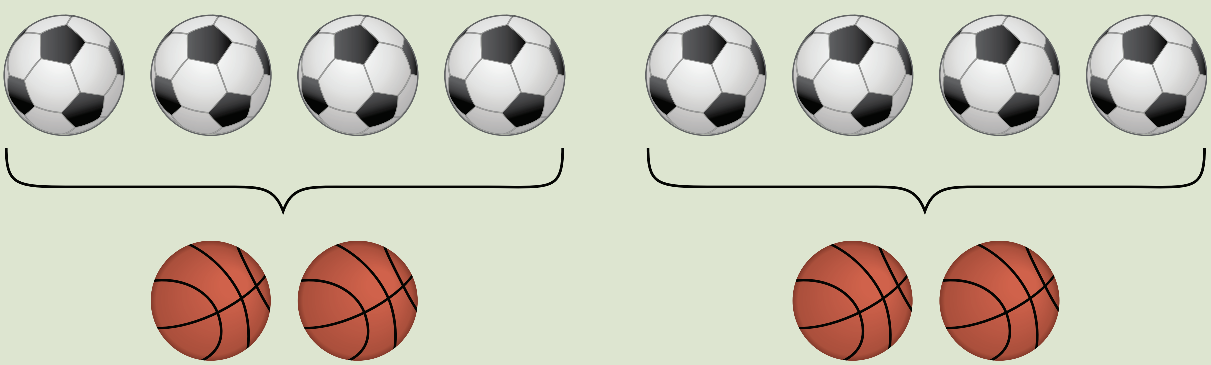
- For example, if the gym has 4 soccer balls for every 2 basketballs, the ratio of soccer balls to basketballs is 4 : 2. That means for every 2 soccer balls, there must be 1 basketball.



4 : 2 = 2 : 1

But that same ratio could also be described as 8 : 4, since for every 8 soccer balls, there would be 4 basketballs.

4 : 2 = 8 : 4



Notice that each time, the first term is double the second. Since that relationship is the same, the ratios are equivalent.

Just as with fractions, if we multiply the two terms by the same amount, we will have an equivalent ratio.

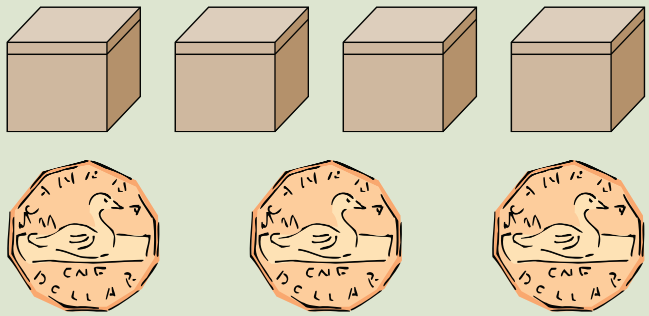
×2 ×3

2 : 1 = 4 : 2 = 12 : 6

The equation that says that two ratios are equal is called a **proportion.**

Rate

- The same is true for rates. A rate of $3 for 4 boxes is the same as a rate of $6 for 8 boxes, $9 for 12 boxes, $1.50 for 2 boxes or 75¢ for 1 box.



The equivalent rate for one item is called a **unit rate**.

Percent

- Percents can also be described in equivalent forms.

  For example, 25% means 25 out of 100, or $\frac{25}{100}$.

  $\frac{25}{100} = \frac{1}{4}$ or any fraction equivalent to $\frac{1}{4}$, for example, 50 out of 200 or 100 out of 400.

When a ratio or percent is written in a form where the two parts have no common factor, the ratio is in **lowest terms**. For example, 50 : 100 is not in lowest terms but the equivalent ratio 1 : 2 is in lowest terms.

1. Which of these ratios are equivalent ratios?

a) 2 : 5 and 2 : 3

b) 2 : 5 and 4 : 10

c) 4 :10 and 6 : 15

2. Draw a picture that explains why the ratio 4 : 5 is equivalent to the ratio 8 : 10. Tell how the picture shows this.

3. Some of the columns in a place value chart are shown:

…	10 000	1000	100	10	1	0.1	0.01	0.001	…

Show that the ratios of **any** column heading to the column heading two columns to its right are equivalent.

4. i) These ratios are equivalent. Fill in the missing terms.

a) 4 : 10 = 2 : ☐

b) 6 : 8 = 9: ☐

c) 8 : ☐ = 20 : 10

d) 52 : 13 = ☐ : 300

e) 3.5 : 10.5 = ☐ : 6

f) 5 : 8 = 1 : ☐

ii) Explain your strategy for part c) and e).

5. One way to compare ratios is to use equivalent ratios. Suppose one dessert uses four cups of strawberries for every three cups of blueberries. Another uses two cups of strawberries for every one cup of blueberries. Which equivalent ratios might you use to decide which is more "strawberry"? Explain your reasoning.

6. The ratio of the number of boys to the number of girls in one class is 7 : 3. The ratio of the number of boys to the number of girls in another class is 3 : 2. Which class has a greater fraction of boys? How do you know?

7. One car drives 32 kilometres every 15 minutes. Another drives 120 kilometres per hour. Use equivalent ratios to decide which is going faster.

8. Five bars of soap cost $3.89. What is an equivalent description of that rate?

9. The heart rates of different animals are shown below.

Dog	Lion	Elephant	Chicken
200 beats in 2 minutes	40 beats in 1 minute	140 beats in 4 minutes	120 beats in 30 seconds

a) Write each rate as a unit rate (number of beats in 1 minute).

b) Which animal's heart beats fastest

10. What fraction with a numerator of 1 or 2 would be good to estimate each percent? Explain your thinking.

a) 30% **b)** 15% **c)** 70% **d)** 11%

11. Canada's population is growing by 1.3% a year.

a) Write 1.3% as an equivalent fraction.

b) Is a growth of 25 people for every 2000 people a higher or lower rate of growth? Explain.

12. Jamila earned 12 marks out of 15 on her project. What would her percentage mark be?

13. Fifty students tried out for a music competition. Twelve were Grade 9 students; ten were Grade 10 students; thirteen were Grade 11 students; and 15 were Grade 12 students. What percent of the students who auditioned were at each grade level?

14. A certain number is 20% of Number A and is also 40% of Number B.

a) Which number is bigger – Number A or Number B?

b) What is the relationship between those numbers?

Solving Ratio and Rate Problems

Learning Goal

- using an equivalent form of a ratio or rate to solve a problem

Open Question

Keena's brother told her these things were true:

a) If you drive 30 kilometres in 13 minutes, then you would drive 32 kilometres in 15 minutes if you kept the same speed.

b) If there are adults and children in a large group and the ratio of the number of adults to the number of children is 5 : 3, then the total number of people has to be a multiple of 8.

c) If you buy $1 Canadian with $1.08 U.S., then you can buy $1 U.S. with 92¢ Canadian.

d) If the length and width of two different rectangles are in a 5 : 2 ratio, the ratio of their diagonals, perimeters, and areas are also in a 5 : 2 ratio.

- With which do you agree? Explain why.

- With which do you disagree? Explain why.

- Make up a similar statement that is true. Prove that it is true.

- Make up a similar statement that sounds true, but really is not true. Prove that it is not true.

Think Sheet

Sometimes a situation is described using a ratio or rate, but we need an equivalent form to be able to solve a problem. For example:

Ratio problem

The Canadian flag's length-to-width dimensions are 2 : 1. We want to know how wide to make a flag that is 51 centimetres long. You want a ratio equivalent to 2 : 1 where the first term is 51.

× ?

We are trying to figure out ☐ if 2 : 1 = 51 : ☐.

- One way to solve the problem is to figure out what we multiplied 2 by to get 51. Then we multiply 1 by the same amount. Since 51 ÷ 2 = 25.5, we multiplied 2 by 25.5 to get 51. Then the width must be 1 × 25.5 centimetres = 25.5 cm.
- Another way to solve the problem is to notice that the width is always half the length, so just take half of 51. This is also 25.5 centimetres.

Rate problem

We know that a family drove 130 kilometres in 1.6 hours. We want to know how far they would travel in 2 hours. We want an equivalent rate where the second value is 2 hours instead of 1.6 hours.

× ?

$$\frac{130}{1.6} = \frac{?}{2}$$

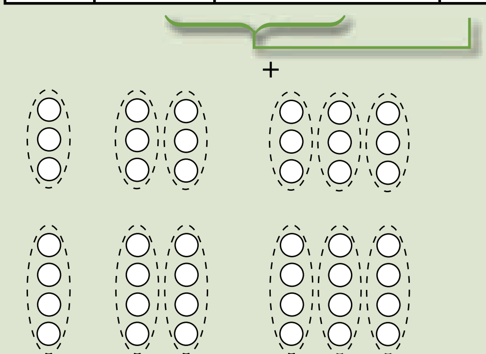
- We could figure out what to multiply 1.6 by to get 2 and then multiply 130 by the same amount.
  2 ÷ 1.6 = 1.25
  1.25 × 130 = 162.5 km
- We could also divide 130 by 1.6 (which is 81.25) to figure out the number of kilometres they drove in one hour, the **unit rate**.

  In two hours, they would go twice as far.
  2 × 81.25 = 162.50 km

- We can build a **ratio table**. A ratio table is a table where equivalent ratios or rates fill the columns. To get from one column to another, we **multiply or divide** both terms by the same amount. We can also **add or subtract** the pairs of terms in two columns to create another equivalent ratio. For example, if both ratios are equivalent to 3 : 4, they both describe groups of 3 items matching groups of 4 items; if you combine them, there are still groups of 3 items matching groups of 4 items.

× 2 × 3

3	6	9	15
4	8	12	20



For the problem about the family driving, we start with a rate with terms 130 and 1.6. We manipulate the values, following the rules above, to try to get a 2 in the bottom row since you want the distance for two hours. There is always more than one way to build a ratio table. One example is:

÷ 8 × 10

Distance	130	16.25	162.5
Time	1.6	0.2	2

÷ 8 × 10

1. A car uses 10 L of gasoline to go 100 kilometres. How much fuel will it use to go these distances?

a) 300 km

b) 450 km

c) 75 km

2. Tara's heart beats 32 times in 24 seconds. What is her heart rate in beats per minute? [Remember: 60 seconds = 1 minute]

3. A competitive runner goes 10 km in about 30 min.

a) What is the unit rate in kilometres per minute?

b) What is the unit rate in minutes per kilometre?

4. Some paint is made by mixing 2 cans of white paint for every 5 cans of red paint. To make the same tint, how many cans of white paint would you need if you used 13 cans of red paint? Explain how you solved the problem.

5. A certain recipe uses 908 g of meat for 8 servings. Figure out the amount of meat you would need for 4, 12, and 15 servings:

a) 4 servings

b) 12 servings

c) 15 servings

6. 6 cans of Brand A soup cost $7.74.

4 cans of Brand B soup cost $5.44.

Describe three different ways to decide which brand is the better buy.

7. A computer can download a file that is 11.6 MB in one second.

a) How long would it take to download a 5 MB file at that same speed?

b) The same computer uploaded an 11.6 MB file in 103 seconds. How long would it take to upload a 5 MB file?

8. Canada's annual birth rate was reported as 10.28 births per 1000 people.

a) If the population of Canada is about 33 million, about how many children are born in a year?

b) If the ratio of male births to female births is 1.06 : 1, about how many of those new babies were boys?

9. A very cautious driver drives a certain distance in 35 minutes driving at a speed of 60 km/h. How much time would he save if he drove that distance at the legal limit of 70 km/h? Explain your thinking.

10. The scale ratio on a map is reported as 1 : 10 000. If two places are 3 cm apart on the map, how far apart are the real places? Describe your answer using two different metric units.

11. The ratio of the perimeter of a certain regular hexagon (six sides equal) to the perimeter of a certain square is 6 : 2. What do you know about the relationship between the side lengths of the two shapes?

12. a) Draw a rectangle with a width to diagonal ratio of 1 : 3. Tell how you did it.

b) Estimate the length to width ratio.

Solving Percent Problems

Learning Goal

- using an equivalent form of a percent to solve a problem

Open Question

Keena's brother told her that these things were true:

a) If you buy something on sale at 10% off and then you get another 15% off, you could have taken 25% off the original price.

b) If you buy something on sale at 10% off and then you pay 13% tax, you could have just added 3% to the original price.

c) If the discount at a store is 35%, you can calculate the sale price by using 65% of the original price.

d) If a smaller number is 80% of a larger one, then the larger one is 120% of the smaller one.

- With which do you agree? Explain why.

- With which do you disagree? Explain why.

- Make up a similar statement that is true. Prove that it is true.

- Make up a similar statement that sounds true, but really is not. Prove that it is not true.

Think Sheet

Percents are ratios where the second term is 100. Sometimes, to solve a problem, we need an equivalent ratio with a different second term.

When We Know the Whole

- For example: Suppose a school has 720 students. At least 60% are required to participate in a fund-raiser before a sponsoring company will help with a school event. We want to know exactly how many students are needed. So instead of the ratio 60 : 100 (60%), we want an equivalent ratio in the form ☐ : 720.

  Solve the problem, using a fraction, decimal or ratio table:

  Using a fraction or a percent

  $60\% = \frac{3}{5}$

  So 60% of $720 = \frac{3}{5}$ of $720 = \frac{3}{5} \times 720$

  $\frac{3}{5} \times 720 = \frac{3 \times 720}{5} = \frac{2160}{5} = 432$

  We can use what we know about simple percents of a number (**benchmarks**) to calculate a more complicated percent.

  For example, since 10% is $\frac{1}{10}$, then 10% of 720 = 72.

  60% would be 6 times as much.

  $6 \times 72 = 432$.

  Using a decimal

  $60\% = \frac{60}{100} = 0.60$

  $0.60 \times 720 = 432$

  Using a Proportion

  Sometimes we solve by writing an equivalent ratio directly.

  If 60 : 100 = ☐ : 720, we divide 720 by 100 to see what we multiplied 100 by to get 72. Then we multiply 60 by that same amount.

  $720 \div 100 = 7.2$

  $60 \times 7.2 = 432$

Using a ratio table

We can build a **ratio table**. A ratio table is a table where equivalent ratios or rates fill the columns. To get from one column to another, we can **multiply or divide** both terms by the same amount. We can also **add or subtract the pairs of terms in two columns** to create another equivalent ratio. For example, if both ratios are equivalent to 3 : 4, they both describe groups of 3 items matching groups of 4 items; if we combine them, there are still groups of 3 items matching groups of 4 items.

We start with the ratio 60 : 100, which is 60%, and build equivalent ratios.

Step 1 It is easier to get to 720 from 20 than is from 100.

÷ 5

60	12		
100	20		720

÷ 5

Step 2 Since 720 is 700 + 20, it would be helpful to get 700 in the bottom row. It is not difficult from 100.

× 7

60	12	420	
100	20	700	720

× 7

Step 3 There is now 700 and 20 in the same row that could be combined to get 720.

+

60	12	420	432	1/5 of sixty + 7 sixties = 7 1/5 sixties
100	20	700	720	1/5 of a hundred + 7 hundreds = 7 1/5 hundreds

+

Note: We could also have multiplied column 2 by 36 (since 36 × 20 = 720) instead of getting the third column that is there. Then 12 × 36 = 432 and we would have had the same result.

- These same strategies can be used even if the percent is more than 100% or a decimal percent: for example, if we wanted 132% of 420, we could multiply by the fraction $\frac{132}{100}$ or decimal 1.32. We could also start a ratio table with a column reading 132 and 100 or we could figure out how $\frac{132}{100} = \frac{\square}{420}$.

When We Know a Part

The same strategies we have been using can be adapted to solve problems in which we know a part and not the whole.

- For example, suppose we know that 15% of the students in a school volunteer at a local soup kitchen. We know that the number of student volunteers is 39 students. We want to know how many students are in the school. Suppose *t* is the total number of students in the school.

Using a decimal

We know that $0.15 \times t = 39$, so we divide 39 by 0.15 to figure out the missing value *t*.

$\frac{39}{0.15} = 260.$

Using a proportion

We can write the **proportion** $\frac{15}{100} = \frac{39}{\square}$. Figure out what we multiplied 15 by to get 39 by dividing 39 by 15. Multiply 100 by the same amount.

$39 \div 15 = 2.6$

$2.6 \times 100 = 260$

Using benchmarks

If we know that 15% is 39, then we know that 5% is $\frac{1}{3}$ of 39, which is 13. That means 10% is 26 (2 × 5%) and 100% is 260 (10 × 10%).

Using a ratio table

We create equivalent ratios with a first term of 39:

÷ 5 × 13

15	3	39
100	20	260

÷ 5 × 13

1. Which of these make sense? Why?

   a) 30% of 58 is about 12.

   b) 74% of 82 is about 60.

   c) 110% of 93 is about 100.

2. Each of these calculations is a step in the solution of a different percent question. Using a percent, what might the question have been?

   a) 0.35×48

   b) $\frac{3}{4} \times 88$

   c) $45 \div 0.2$

3. People often recommend giving a 15% tip for good service. If a meal costs $45.29, what tip would you leave if you wanted to leave about 15%? Explain how to estimate without using a calculator.

4. The Harmonized Sales Tax (HST) is 13%. Suppose an item costs $89 and you have to pay HST.

   a) How might you estimate the HST on the item before calculating it?

   b) What is the HST on that item?

   c) How much is the total cost including the tax?

5. Anika calculated the price for an item including HST by multiplying by 1.13. Why does that make sense?

6. Jayda bought a sweater on sale. The discount was 30%. Before the tax was added, the sale price of the sweater was $34.97. Figure out the original price and describe the strategy you used.

7. Ethan had $550 in the bank. He took out $50 to spend on a present.

   a) What percent of his money is still in the bank?

   b) How much could he have taken out if he wanted to make sure that 70% was still in the bank?

8. The side length of one square is 112.5% of the side length of another. What percent of the area of the small square is the area of the larger one? Explain why your answer seems reasonable.

9. A new employee in a company earns only 63% the salary of a more experienced employee. If the experienced employee earns $57 000, what would be the salary of the new employee?

10. A town of 4827 people is expected to grow by 3.2% next year.

   a) What is the expected population for the next year?

   b) What percent of the old population is the new one?

   c) What percent of that new population is the old one?

11. What is the same and what is different about how you solve these two problems?

   A: I spent $40 and it was 24% of what I had. How much did I have?

   B: I spent 24% of the $40 I had. What did I spend?

10-Part Spinner