GAP CLOSING

Proportional Reasoning

Intermediate / Senior
Facilitator's Guide

Topic 4

Proportional Reasoning

The Facilitator's Guide for this entire module, including the Diagnostic plus all Intervention Materials, can be downloaded at http://www.edugains.ca/resources/LearningMaterials/GapClosing/Grade9/4-PropReasoning_FG_IS.pdf. The Student Book for this module can be downloaded at http://www.edugains.ca/resources/LearningMaterials/GapClosing/Grade9/4-PropReasoning_SB_IS.pdf.

PROPORTIONAL REASONING

Relevant Expectations for Grade 9

MPM1D

Number Sense and Algebra

- solve problems requiring the manipulation of expressions arising from applications of percent, ratio, rate, and proportion
- solve problems that can be modelled with first-degree equations, …

Linear Relations

- identify, through investigation, some properties of linear relations and apply these properties to determine whether a relation is linear or non-linear
- compare the properties of direct variation and partial variation in applications, ….

Analytic Geometry

- determine, through investigation, various formulas for the slope of a line segment or a line, and use the formulas to determine the slope of a line segment or a line
- determine the meaning of the slope and y-intercept for a linear relation arising from a realistic situation ….

Measurement and Geometry

- solve problems involving the areas and perimeters of composite two-dimensional shapes

MPM1P

Number Sense and Algebra

- illustrate equivalent ratios, using a variety of tools
- represent, using equivalent ratios and proportions, directly proportional relationships arising from realistic situations
- solve for the unknown value in a proportion, using a variety of methods
- make comparisons using unit rates
- solve problems involving ratios, rates, and directly proportional relationships in various contexts using a variety of methods
- solve problems requiring the expression of percents, fractions, and decimals in their equivalent forms

Linear Relations

- identify, through investigation, some properties of linear relations and apply these properties to determine whether a relation is linear or non-linear
- determine, through investigation, that the rate of change of a linear relation can be found by choosing any two points on the line that represents the relation, finding the vertical change between the point and the horizontal change between the points and writing the ratio $\frac{\text{rise}}{\text{run}}$
- compare the properties of direct variation and partial variation in applications, ….

Measurement and Geometry

- solve problems involving the areas and perimeters of composite two-dimensional shapes

Possible reasons why a student might struggle with ratios, rates, and percents

Students may struggle when solving problems involving ratios, rates, and percents.

Some of the problems include:

- comparing numbers additively rather than multiplicatively, e.g., believing that the ratio 4 : 6 is equivalent to the ratio 6 : 8 since you added 2 both times
- difficulty justifying why two ratios or rates are equivalent other than by describing mechanical procedures
- confusing the various ratios involved in a single situation. For example, if the ratio of the number of boys to the number of girls in a class is 3 : 4, the student might think that the class is 75% boys. The student is using the part-part ratio instead of the part-whole ratio.
- lack of understanding that solving a ratio, rate, or percent problem always involves determining an equivalent ratio in a preferred form for that particular situation
- difficulty solving a percent problem when the whole is the unknown, e.g., a student is able to calculate 30% of 50 but has difficulty calculating the number for which 15 is 30%.
- inability to draw pictures to model a ratio, rate, or percent situation to help when a solution is not obvious
- inability to determine an equivalent ratio when the terms are not whole numbers or when the terms of one ratio are not integer multiples of the terms of the other
- lack of comfort with the notion of what a percent greater than 100% means
- difficulty distinguishing between a percent of and a percent change
- difficulty dealing with decimal percents, e.g., thinking that 0.5% of 20 is 10

DIAGNOSTIC

Materials
• calculator

Administer the diagnostic

Using diagnostic results to personalize interventions

Intervention materials are included on each of these topics:
• describing and representing ratios, rates, and percents
• equivalent forms of rates, ratios, and percents
• solving ratio and rate problems
• solving percent problems

You may use all or only part of these sets of materials, based on student performance with the diagnostic. If students need help in understanding the intent of a question in the diagnostic, you are encouraged to clarify that intent.

Evaluating Diagnostic Results	Suggested Intervention Materials
If students struggle with Questions 1–6	use *Describing and Representing Ratios, Rates and Percents*
If students struggle with Questions 7–9	use *Equivalent Forms of Rates, Ratios and Percents*
If students struggle with Questions 10–13	use *Solving Ratio and Rate Problems*
If students struggle with Questions 14–17	use *Solving Percent Problems*

Solutions

1. **a)** 3 : 8 **b)** 8 : 11 **c)** e.g., 4 : 7

2. **a)** Vada **b)** No, e.g., 8 : 4 would be twice as tall, not 5 : 4

3. e.g., She goes 0.18 kilometres every minute.

4. **a)** 23% **b)** 77%

5. **a)** DOES NOT MAKE SENSE **b)** MAKES SENSE
c) DOES NOT MAKE SENSE

6. e.g., 35% is more than one fourth so it would mean more than 1 person in 4 is an adult; it would be like having one teacher for every 3 students and that does not make sense even if you include administrative staff and custodians.

7. **a)** 4 **b)** 4 **c)** 20

8. 360 times

9. **a)** e.g., $\frac{40}{100}$ **b)** e.g., $\frac{112}{100}$ **c)** e.g., $\frac{7}{200}$

10. **a)** $5.22 **b)** $6.96

11. 104 km

12. about \$1.15

13. 30%

14. \$9.09

15. a) FALSE **b)** TRUE **c)** TRUE

16. e.g., 10% of 120 is 12, so 40% is 48, not 30.

17. \$37.50

Diagnostic

1. There are 8 boys and 3 girls on the Tech Team.

a) Write the ratio of he number of girls to number of boys in the form ☐ : ☐.

b) Write the ratio of the number of boys to the number on the whole team.

c) Another Tech Team of 11 students has a higher ratio of number of girls to number of boys. What could the ratio be?

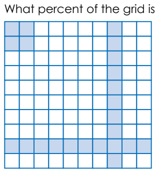
2. The ratio of Vada's height to Melissa's height is 5 : 4.

a) Who is taller?

b) Is she twice as tall? How do you know?

3. Valene's running rate is 0.18 km/min. Explain what that means.

4. a) What percent of the grid is shaded?



b) What percent is not shaded?

Diagnostic (Continued)

5. Indicate whether each statement does or does not make sense by circling your choice.

a) 8% of something is a lot of it.

MAKES SENSE DOES NOT MAKE SENSE

b) 80% of something is a lot more than half of it.

MAKES SENSE DOES NOT MAKE SENSE

c) 35% of the people in a high school building on a school day are adults, not students.

MAKES SENSE DOES NOT MAKE SENSE

6. Explain your answer to Question 5c.

7. Complete the missing amounts so that the ratios are equivalent.

a) 2 : 7 = ☐ : 14 **b)** 5 : 10 = ☐ : 8 **c)** 12 : ☐ = 3 : 5

8. Suppose your heart beats 144 times in 2 minutes. How many times would you expect it to beat in 5 minutes?

9. What fraction is equivalent to each percent?

a) 40% **b)** 112% **c)** 3.5%

10. Three bars of soap cost $2.61. At this rate, how much would each number of bars below cost?

a) 6 bars

b) 8 bars

Diagnostic (Continued)

11. A car goes 78 km in 45 minutes. At that speed, how far would it go in an hour?

12. A 2.6 L container of juice costs $3.00. How much are you paying for 1 L?

13. Suppose the ratio of the number of boys to the number of girls in a class is 7 : 3. What percent of the class is girls?

14. A T-shirt is priced at $12.99. The store is offering a discount of 30%. How much will the shirt cost (before taxes)?

15. Tell if each statement is TRUE or FALSE by circling the correct word.

a) 40% of 120 is about 30. TRUE FALSE

b) 20% of 83 is about 16. TRUE FALSE

c) 11% of 198 is about 20. TRUE FALSE

16. Explain your answer to Question 15a.

17. Lea spent $25 of the money she saved. She still has 60% of her money left. How much does she have left?

USING INTERVENTION MATERIALS

The purpose of the suggested work is to help students build a foundation for successfully working with proportions.

Each set of intervention materials includes a single-task Open Question approach and a multiple-question Think Sheet approach. These approaches both address the same learning goals, and represent different ways of engaging and interacting with learners. You could assign just one of these approaches, or sequence the Open Question approach before, or after the Think Sheet approach.

Suggestions are provided for how best to facilitate learning before, during, and after using your choice of approaches. This three-part structure consists of:

- Questions to ask before using the approach
- Using the approach
- Consolidating and reflecting on the approach

Describing and Representing Ratios, Rates, and Percents

Learning Goal

- representing comparisons based on multiplying as either ratios, rates and percents

Open Question

Materials
- Internet access

Questions to Ask Before Using the Open Question

◊ *What does it mean if someone says the ratio of the number of boys to the ratio of the number of girls in a class is 3 : 4?* (e.g., It means that there are 3 boys for every 4 girls.)
◊ *What other information do you know based on that?* (e.g., There are more girls than boys; that for every 7 students, 3 are boys and 4 are girls; and that the ratio of the number of girls to the number of boys is 4 : 3.)
◊ *Why might you want to know a ratio like that?* (e.g., Two teachers want to compare ways their classes are alike and different and this is one way.)
◊ *Why do people call ratios comparisons?* (e.g., Ratios tell you how much of one thing is being compared to how much of another.)
◊ *Why is a percent a ratio?* (e.g., It is comparing a number to 100. For example 50% means 50 compared to 100.)
◊ *"I drove 30 kilometres in one hour." How is this a comparison?* (e.g., You are comparing the distance you travel to the amount of time it takes to travel that distance.)

Using the Open Question

Students might focus on only one of the three topic choices or all of them.

Support students in using the board's safe computer use policy.

By viewing or listening to student responses, note if they:
- can distinguish between ratios and rates;
- recognize that ratios, rates, and percents all represent comparisons;
- can describe what two things are being compared with a ratio, rate, or percent.

Consolidating and Reflecting on the Open Question

◊ *Why did you say that a batting average is a ratio?* (e.g., It compares the number of hits a person gets to the number of times they could get a hit.)
◊ *How are your rates different from your ratios?* (e.g., The ratios use the same units for both terms but the rates do not.)
◊ *What would a high rate for pollution mean?* (e.g., There are more particles of bad things in a given amount of water or air.)
◊ *Do you think that all prices are rates?* (e.g., Yes, since you are comparing the amount of money for the number of packages or the mass or the volume.)

Solutions

e.g.,

ratio

- A batting average in baseball is a ratio where the second term is 1000. It is a way of comparing how many hits a player gets to the total chances of getting a hit.

  http://www.ehow.com/how_9730_calculate-batting-average.html

- There is something called an assist-to-turnover ratio in basketball. If this ratio is high, the player is good at passing to other players who make shots and also good at keeping the ball away from the other team. It compares the number of assists to the number of turnovers.

http://www.ehow.com/how_2092831_calculate-assists-turnover-ratio-basketball.html

- In football, there is a fumble ratio that compares the number of fumbles to 100 touches of the ball. It tells how 'awkward' a player is with the ball.

  http://www.pro-football-reference.com/blog/?p=522

rate

- In 2002, it was reported that the air pollution rate in Cairo was 20 times the acceptable level. This rate compares the Cairo rate of air pollution to the world average.

  http://aeamisr.org/news/air-pollution-in-cairo/

- Toronto public health estimated that 440 deaths per year were due to traffic pollution. This rate compares the number of deaths to the time period of one year.

  http://www.toronto.ca/health/hphe/pdf/air_pollution_burden_boh.pdf

- The City of Calgary charges $95 a tonne to dump waste at their landfills. This rate compares money to mass.

  http://www.calgary.ca/UEP/WRS/Pages/Landfill-information/Landfill-Rates.aspx

percent

- In New York City, there is a Percent for Art program; 1% of the budget for new buildings is spent on art. 1% compares the percent spent on art to the whole budget for new buildings.

  http://www.nyc.gov/html/dcla/html/panyc/panyc.shtml

- In 2007, some people who support the arts in the United States were hoping for a 40% increase in spending on the arts. They are comparing the amount of increase they want to the money already being spent.

  http://www.bloomberg.com/apps/news?pid=newsarchive&sid=ajxh6929.2Yk

- In New York City, funding for arts supplies and musical instruments fell 68% between 2006 and 2009. That percent compares the loss to the original whole.

  http://articles.nydailynews.com/2010-07-01/local/29437363_1_arts-education-arts-classes-full-time-arts-teachers

Think Sheet

Materials
- counters
- blank hundredth grids
- rulers

Questions to Ask Before Assigning the Think Sheet

◊ *What does it mean if someone says the ratio of the number of classical songs played on a radio station to the number of pop songs is 1 : 4?* (e.g., It means that for every one classical song played, there are four pop songs played.)
◊ *What other information do you know based on that ratio?* (e.g., There are more pop songs played than classical ones; for every five songs played, four are pop and one is classical; and the ratio of the number of pop songs to the number of classical songs is 4 : 1)
◊ *Why might you want to know a ratio like that?* (e.g., To decide if this is the kind of radio station I want to listen to.)
◊ *Why do people call ratios comparisons?* (e.g., A ratio tells how much of one thing is being compared to how much of another.)
◊ *Why is a percent a ratio?* (e.g., It is comparing a number to 100, for example, 50% means 50 compared to 100.)
◊ *"I drove 30 kilometres in one hour." How is this a comparison?* (e.g., You are comparing the distance you travel to the amount of time it takes to travel that distance.)

Using the Think Sheet

Read through the introductory box with the students and make sure they understand the material explained in the instructional box.

Encourage students to use counters to model the boy/girl ratio shown.

Provide rulers for Question 3 and blank hundredths grids for Question 7.

Assign the tasks.

By viewing or listening to student responses, note if they:
- can compare two ratios;
- can interpret a ratio;
- recognize how different ratios can describe the same situation;
- recognize that the term *per* usually connotes a ratio or rate;
- can model and interpret basic percents.

Consolidating and Reflecting: Questions to Ask After Using the Think Sheet

◊ *Why did you say a 1 : 2.5 ratio of water to orange juice is weaker than a 1 : 4 ratio?* (e.g., There is 1 cup of water in a 5 cup mixture for the 1 : 4 ratio, so the water is only $\frac{1}{5}$ of the mixture, but there is 1 cup of water in a 3.5 cup ratio for the 1 : 2.5 mixture, so the water is more than $\frac{1}{5}$ of the mixture.)
◊ *Why was it important to know that the ratio 3 : 2 in Question 2 described the model lengths first and then the real lengths to decide whether the model was, or was not, bigger than the bird?* (e.g., If the model was smaller, it would have been 3 : 2 to compare real lengths to model lengths.)
◊ *What length-to-width ratios of a rectangle would be squares or almost squares?* (e.g., 1 : 1 or maybe 1.1 : 1.) *Why?* (e.g., The length and width should be almost the same.)
◊ *Why does the word "*per*" suggest a comparison?* (e.g., You are comparing the item before the word *per* to the item after the word.)
◊ *What percent do you think of as a lot? A little?* (e.g., 90% is a lot, since it is most of something; 5% is a little, since it is only 5 parts out of a whole 100.)

Solutions

1. a) ◯◯ □□□
b) ◯◯◯◯ □

2. C; 2 out of 6 is definately less red than 2 out of 7. 2 out of 6 is like 1 out of 3 or 3 out of 9 which is less than 3 out of 8.

3. 1 : 2.5, since you will use 1 cup of water in only 3.5 cups compared to 1 cup in larger amounts.

4. **a)** bigger, since the model would use 3 centimetres for every real 2 centimetres
b) 3 cm
c) A length on the actual bird is compared to the matching length on the model.

5. **a)** No, e.g., If it were, the ratio would be close to 4 : 4, not 12 : 4.
b) 12 : 32
c) e.g., When you have a ratio, the units have to be the same; 12 metres is actually 1200 centimetres, so the ratio would be 1200 : 4, not 12 : 4.

6. 3 : 10, The fat is a smaller number compared to the whole body weight.

7. e.g., A pulse tells how many heartbeats per minute, so it's comparing number of heartbeats to 1 minute.

8. e.g., grams per dollar; average number of children per family; number of doctors per 1000 people in the population

9. **a)** 50% **b)** 60% **c)** 25%
d) e.g., 90% **e)** e.g., 8%

10. **a)** 95% **b)** 95 : 5

11. **a)** It does not make sense; e.g., that would mean I would be exercising 8 hours a day and I do not do that.
b) It makes sense since; e.g., 50% means half, and since heads and tails are equally likely, you have half a chance for heads.
c) It makes sense; e.g., 1% is 1 out of 100 and there are 100 pennies in a loonie.
d) It does not have to make sense; e.g., It could have been 5% off a really expensive sweater and 10% off an inexpensive one.

Open Question

Describing and Representing Ratios, Rates, and Percents

Learning Goal

- representing comparisons based on multiplying as either ratios, rates and percents

Open Question

Ratios, rates, and percents all describe comparisons.

For example:

Ratio – A recipe uses 3 parts flour for every 1 part sugar.

Rate – A painter uses one can of paint to cover 2 walls in 5 hours.

Percent – 52¢ is 52% of 100.

- Search the Internet and find three or four examples for each type of comparison (ratio, rate, and percent) that are related to one of these topics:
  - environmental issues
  - sports
  - the arts
- Each time, indicate what two things are being compared. Write down the url.

Think Sheet

Describing and Representing Ratios, Rates, and Percents (Cont.)

Think Sheet

These expressions describe comparisons using a ratio, a rate and a percent:

- Three girls for every four boys is a **ratio** that compares the proportion of girls to boys.

  We can write that ratio as 3 : 4.
  3 and 4 are called **terms** in the ratio: 3 is the first term and 4 is the second term.

  For example, if a class has 12 girls and 16 boys, you could arrange them to show that there are 3 girls for every 4 boys.

  G G G B B B B [3 girls for 4 boys]
  G G G B B B B [3 girls for 4 boys]
  G G G B B B B [3 girls for 4 boys]
  G G G B B B B [3 girls for 4 boys]
  12 girls 16 boys

  Notice that the ratio of girls to all the students is not 3 : 4; it is 3 : 7 since there are 7 students in total for every 3 girls. We could also say $\frac{3}{7}$ of the class is girls.

- 3 boxes for $4 is a **rate** that compares an amount of goods to a dollar amount.

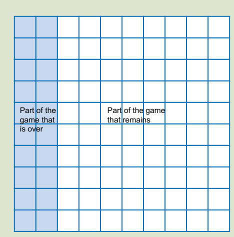
  For example, if 3 boxes of one brand of a product costs $4 and 3 boxes of another brand of that product costs $5, we could choose the best buy by comparing the prices of 1 box of each brand or by comparing how much of each brand $1 buys.

  A rate compares two things measured in different units. This time the units were boxes and money. Sometimes rates describe speed (kilometres per hour or metres per second) or a map scale (1 centimetre on the map for every 12 kilometres of real distance.)

Describing and Representing Ratios, Rates, and Percents (Cont.)

- Saying that 20% of the game is over uses a **percent** to compare a part of the game to the whole game. A percent is a ratio where the second term is 100.

  We could also write 20% as the ratio 20 : 100.
  Since percents are ratios out of 100, a 100-grid is a good way to represent a percent. The model shows 20% (20 squares out of 100).



  To compare the part of the game that is over to the part that remains, you could use the ratio 20 : 80 and use the same picture.

1. Draw a picture to show each ratio.

a) 2 circles : 3 squares **b)** 4 circles : 5 shapes

2. Which circle is more red? Explain why.

A: 2 parts red out of 6 equal parts.

B: 2 parts red out of 7 equal parts

C: 3 parts red out of 8 equal parts

Describing and Representing Ratios, Rates, and Percents (Cont.)

3. You can mix 1 cup of water with different numbers of cups of orange juice to get different tastes. Which of these ratios of water to orange juice will taste the most watery? Explain why.

1 : 3 1 : 4 1 : 2.5 1 : 3.5

4. Yasir built a scale model of a bird. He decided to use a ratio of:

3 : 2
model lengths : real lengths

a) Was the model bigger or smaller than the real bird? Explain.

b) If a claw on the bird was really 2 centimetres, how long was it on the model?

c) What does the ratio 2 : 3 tell in this situation?

5. The ratio of the length to the width of a rectangle is 12 : 4.

a) Is the rectangle almost square or not? Explain.

b) What is the ratio of the length to the perimeter?

c) Why might the length and width be either 12 centimetres and 4 centimetres or 12 metres and 4 metres, but not 12 metres and 4 centimetres?

Describing and Representing Ratios, Rates, and Percents (Cont.)

6. One measure of fitness is based on comparing your body fat mass to your total mass. A low ratio suggests that you are more fit. Which ratio of body fat mass to total mass is better: 3 : 10 or 3.4 : 10? Explain.

7. Why does your pulse describe a rate?

8. The word ***per*** is often used to describe rates. For example, you might talk about kilometres *per* hour. It can also be shown as a / (e.g., km/hr). List at least three other rates you might describe using the word *per*.

9. What percent might you be representing on a 100-grid if you shade:

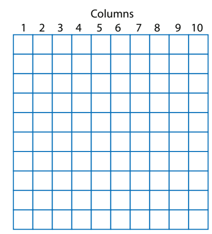
a) every other square?

b) columns 1, 2, 4, 5, 7, and 8?

c) every 4th square?

d) most, but not all, of the squares?

e) just a few squares here and there?



Describing and Representing Ratios, Rates, and Percents (Cont.)

10. On a particular day, 5% of all of the people in a school building are adults. The rest are students.

a) What percent are not adults?

b) What is the ratio of students to adults?

11. Which of these statements make sense? Explain your reasoning for each one.

a) You exercise vigorously 30% of the day.

b) If you flip a coin, it will land on heads 50% of the time.

c) A penny is worth 1% of a loonie.

d) If you buy a sweater and save 5%, you must have saved less than your friend who bought a sweater at 10% off.

Equivalent Forms of Ratios, Rates, and Percents

Learning Goal

- representing comparisons based on multiplying in a variety of different ways

Open Question

Materials
- Spinner template paper clip and pencil (for spinner)
- calculators

Questions to Ask Before Using the Open Question

◊ *Why would someone say that $\frac{2}{3}$ and $\frac{4}{6}$ are equivalent fractions?* (e.g., They are equal.) *What does that mean?* (e.g., If you shaded parts of a whole to show each fraction, they would look the same.)
◊ *What if you were thinking about $\frac{2}{3}$ of a group and $\frac{4}{6}$ of a group?* (e.g., It might mean that 2 out of every 3 counters are a certain colour, but that is the same as saying 4 out of 6 are that colour — 2 out of the first 3 and 2 out of the second 3.)
◊ *Why might you want to know about equivalent fractions?* (e.g., If you want to know how many students are boys when you know that $\frac{2}{3}$ of a set of 18 students are boys, you would want to think of $\frac{2}{3}$ as $\frac{12}{18}$.)
◊ *What are equivalent ratios?* (e.g., Two ratios that give you the same information, such as 2 out of 3 and 4 out of 6.)

Using the Open Question

Students spin the spinner 9 times to fill in all the required digits. Allow them to either spin all nine times and then put in the digits wherever they want or spin, immediately place a digit, and repeat eight times. If they are uncomfortable using a 0 as a tens digit, allow them to spin again.

Make sure that they understand that in all three cases, the 10 could become the first or second term of an equivalent ratio or rate.

By viewing or listening to student responses, note if they:
- have a strategy to determine an equivalent ratio or rate;
- use number sense to decide if one term is easier to change to a 10 than another;
- realize that percents are ratios and that an equivalent ratio would not have the number 100 as the second term.

Note: You might introduce the word *proportion* to refer to a statement that indicates that two ratios are equivalent or equal.

Consolidating and Reflecting on the Open Question

◊ *How did you figure out that 43 : 48 is equivalent to 10 : 11.16?* (e.g., I divided 43 by 4.3 to get 10, so I also divided 48 by 4.3.) *Why does the second ratio make sense?* (e.g., Both times, the first number is somewhat less than the second one.)
◊ *How else could you have written 43 : 48 as an equivalent ratio with a 10?* (e.g., I could have divided by 4.8 and then the second term would have been 10.)
◊ *When you wrote 36% as an equivalent ratio, what did you do?* (e.g., I know that 36% means 36 : 100, so I divided both numbers by 10.) *Why is it not a percent anymore?* (e.g., It is out of 10 and not out of 100.)
◊ *If you had a rate of 34 km/5 hours, which term would you change to 10? Why?* (e.g., I would change the second one since I could just double both numbers and that is quite easy.)

Solutions

e.g.,

43 : 48	21 km/8h	47%
10 : 11.16	26.25 km/10 h	4.7 : 10
71 : 43	25 km/3 h	28%
10 : 6.06	83.3 km/10 h	2.8 : 10
55 : 36	29 km/4 h	36%
15.28 : 10	10 km/1.38 h	10 : 27.78

Think Sheet

Materials
- calculators
- blank hundredths grids

Questions to Ask Before Assigning the Think Sheet

◊ *Why would someone say that $\frac{2}{3}$ and $\frac{4}{6}$ are equivalent fractions?* (e.g., They are equal.) *What does that mean?* (e.g., If you shaded part of a whole to show each fraction, they would look the same.)
◊ *Why might you want to know about equivalent fractions?* (e.g., If you want to know how many students are boys when you know that $\frac{2}{3}$ of a set of 18 students are boys, you would want to think of $\frac{2}{3}$ as $\frac{12}{18}$.)
◊ *What are equivalent ratios?* (e.g., Two ratios that give you the same information, such as 2 out of 3 and 4 out of 6.)

Using the Think Sheet

Read through the introductory box with the students and make sure they understand the material explained in the instructional box.

Assign the tasks.

By viewing or listening to student responses, note if they:
- have a strategy to determine an equivalent ratio or rate;
- can use a model to show why ratios are equivalent;
- recognize familiar mathematical situations that involve ratios;
- can compare ratios or rates;
- can describe a rate as a unit rate;
- relate percents and fractions.

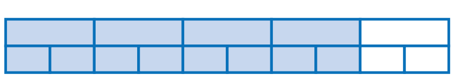
Consolidating and Reflecting: Questions to Ask After Using the Think Sheet

◊ *What sort of picture did you draw to show that 4 : 5 is equivalent to 8 : 10?* (e.g., I drew 4 squares and 5 circles and did it twice. Then there were 8 squares and 10 circles but there were still 4 squares for each 5 circles.)
◊ *Why might it be useful to notice that 20 is twice 10 to figure out the answer to Question 4c?* (e.g., I know that the first term has to be double the second, so that the missing term has to be half of 8, which is 4.)
◊ *How did you compare the ratios in Question 6?* (e.g., I know that a ratio of 7 : 3 of the number of boys to the number of girls means that $\frac{7}{10}$ of the students are boys. But a ratio of 3 : 2 means $\frac{3}{5}$ are boys and I know that $\frac{7}{10}$ is more than $\frac{3}{5}$.)
◊ *Why are unit rates useful?* (e.g., You can tell really quickly which rate is greater.)
◊ *Why are percents useful equivalents to other ratios or to fractions?* (e.g., To compare percents, you use what you know about whole numbers.)

Solutions

1. b, c

2. e.g.,



The picture shows that having 4 parts out of 5 dark is really the same as having 8 out of 10 dark, if you use smaller sections.

3. e.g., 1000 : 10 = 100 : 1
10 000 : 100 = 100 : 1
10 : 0.1 = 100 : 1
1 : 0.01 = 100 : 1

I tried it with some of the columns that were there and it worked each time.

I realize that in a place value chart, each column is worth 10 of the one to its right. That means if you go one more column left, it is worth 10 × 10 = 100 of the column you started with, no matter where it is on the chart.

4. **a)** 5 **b)** 12 **c)** 4
d) 1200 **e)** 2 **f)** 1.6

For part c, I know that 20 : 10 is the same as 2 : 1 and so I needed half of 8 which is 4.

For part e, I noticed that 10.5 is 3 times as much as 3.5, so I divided 6 by 3. That is 2.

5. e.g., 4 : 3 and 4 : 2

6. The first class; e.g., 7 : 3 means 7 out of 10 students are boys.

3 : 2 means 3 out of every 5, which is 6 out of 10, not 7 out of 10.

7. 32 km/15 min is the same as 128 km/h and that is faster than 120 km/h.

8. e.g., 10 bars for $7.78

9. **a)** dog: 100 beats/min
lion: 40 beats/min
elephant: 35 beats/min
chicken: 240 beats/min
b) chicken

10. **a)** e.g., $\frac{1}{3}$, I know that 3 × 33 = 99, which is almost 100, so $\frac{1}{3}$ is about 33%, which is close to 30%.

b) e.g., $\frac{1}{5}$, I know that 20% = $\frac{1}{5}$, so 20 × 5 = 100 and I think 15% is close.

c) e.g., $\frac{2}{3}$, I know that $\frac{1}{3}$ = 33% and $\frac{2}{3}$ = 67% and 70 is close to 67.

d) e.g., $\frac{1}{10}$, I know that 11% = $\frac{11}{100}$. That is close to $\frac{10}{100} = \frac{1}{10}$.

11. **a)** e.g., $\frac{13}{1000}$ **b)** lower rate since $\frac{26}{2000}$ is higher than $\frac{25}{2000}$.

12. 80%

13. Grade 9 was 24%
Grade 10 was 20%
Grade 11 was 26%
Grade 12 was 30%

14. **a)** Number A
b) Number A is twice Number B

Open Question

Equivalent Forms of Rates, Ratios, and Percents

Learning Goal

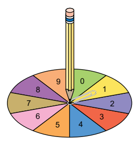
- representing comparisons based on multiplying in a variety of different ways

Open Question

Any fraction can be written in equivalent forms.

For example, $\frac{2}{3} = \frac{6}{9}$, $\frac{9}{15} = \frac{3}{5}$, and $\frac{4}{100} = 0.04$.

Ratios, rates, and percents can also be written in equivalent forms.



- Spin the spinner 9 times to fill in the digits.

Ratio	Rate	Percent
☐☐:☐☐	☐☐km/☐h	☐☐%

- Show that each can be written in an equivalent form that somewhere includes the number 10.

- Repeat twice more.

Ratio	Rate	Percent
☐☐:☐☐	☐☐km/☐h	☐☐%
☐☐:☐☐	☐☐km/☐h	☐☐%

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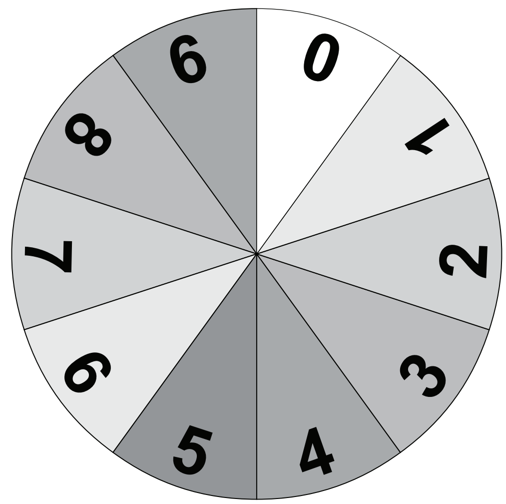
10-Part Spinner



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Think Sheet

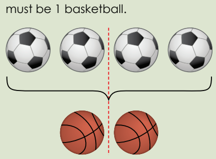
Equivalent Forms of Rates, Ratios, and Percents (Continued)

Think Sheet

There are many ways to describe the same ratio or rate. The **equivalent forms** can be other ratios or rates, or they can be fractions or percents.

Ratio

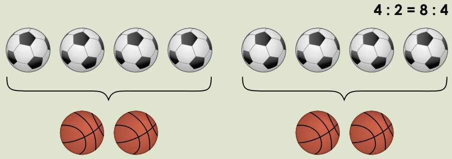
- For example, if the gym has 4 soccer balls for every 2 basketballs, the ratio of soccer balls to basketballs is 4 : 2. That means for every 2 soccer balls, there must be 1 basketball.



4 : 2 = 2 : 1

But that same ratio could also be described as 8 : 4, since for every 8 soccer balls, there would be 4 basketballs.

4 : 2 = 8 : 4



Notice that each time, the first term is double the second. Since that relationship is the same, the ratios are equivalent.

Just as with fractions, if we multiply the two terms by the same amount, we will have an equivalent ratio.

×2 ×3

2 : 1 = 4 : 2 = 12 : 6

The equation that says that two ratios are equal is called a **proportion.**

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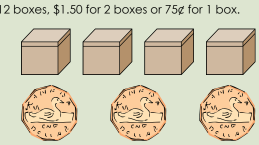
Equivalent Forms of Rates, Ratios, and Percents (Continued)

Rate

- The same is true for rates. A rate of \$3 for 4 boxes is the same as a rate of \$6 for 8 boxes, \$9 for 12 boxes, \$1.50 for 2 boxes or 75¢ for 1 box.



The equivalent rate for one item is called a **unit rate**.

Percent

- Percents can also be described in equivalent forms.

  For example, 25% means 25 out of 100, or $\frac{25}{100}$.

  $\frac{25}{100} = \frac{1}{4}$ or any fraction equivalent to $\frac{1}{4}$, for example, 50 out of 200 or 100 out of 400.

When a ratio or percent is written in a form where the two parts have no common factor, the ratio is in **lowest terms**. For example, 50 : 100 is not in lowest terms but the equivalent ratio 1 : 2 is in lowest terms.

1. Which of these ratios are equivalent ratios?

a) 2 : 5 and 2 : 3

b) 2 : 5 and 4 : 10

c) 4 :10 and 6 : 15

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Equivalent Forms of Rates, Ratios, and Percents (Continued)

2. Draw a picture that explains why the ratio 4 : 5 is equivalent to the ratio 8 : 10. Tell how the picture shows this.

3. Some of the columns in a place value chart are shown:

…	10 000	1000	100	10	1	0.1	0.01	0.001	…

Show that the ratios of **any** column heading to the column heading two columns to its right are equivalent.

4. i) These ratios are equivalent. Fill in the missing terms.

a) 4 : 10 = 2 : ☐ **b)** 6 : 8 = 9: ☐ **c)** 8 : ☐ = 20 : 10

d) 52 : 13 = ☐ : 300 **e)** 3.5 : 10.5 = ☐ : 6 **f)** 5 : 8 = 1 : ☐

ii) Explain your strategy for part c) and e).

5. One way to compare ratios is to use equivalent ratios. Suppose one dessert uses four cups of strawberries for every three cups of blueberries. Another uses two cups of strawberries for every one cup of blueberries. Which equivalent ratios might you use to decide which is more "strawberry"? Explain your reasoning.

Equivalent Forms of Rates, Ratios, and Percents (Continued)

6. The ratio of the number of boys to the number of girls in one class is 7 : 3. The ratio of the number of boys to the number of girls in another class is 3 : 2. Which class has a greater fraction of boys? How do you know?

7. One car drives 32 kilometres every 15 minutes. Another drives 120 kilometres per hour. Use equivalent ratios to decide which is going faster.

8. Five bars of soap cost $3.89. What is an equivalent description of that rate?

9. The heart rates of different animals are shown below.

Dog	Lion	Elephant	Chicken
200 beats in 2 minutes	40 beats in 1 minute	140 beats in 4 minutes	120 beats in 30 seconds

a) Write each rate as a unit rate (number of beats in 1 minute).

b) Which animal's heart beats fastest

Equivalent Forms of Rates, Ratios, and Percents (Continued)

10. What fraction with a numerator of 1 or 2 would be good to estimate each percent? Explain your thinking.

a) 30% **b)** 15% **c)** 70% **d)** 11%

11. Canada's population is growing by 1.3% a year.

a) Write 1.3% as an equivalent fraction.

b) Is a growth of 25 people for every 2000 people a higher or lower rate of growth? Explain.

12. Jamila earned 12 marks out of 15 on her project. What would her percentage mark be?

13. Fifty students tried out for a music competition. Twelve were Grade 9 students; ten were Grade 10 students; thirteen were Grade 11 students; and 15 were Grade 12 students. What percent of the students who auditioned were at each grade level?

14. A certain number is 20% of Number A and is also 40% of Number B.

a) Which number is bigger – Number A or Number B?

b) What is the relationship between those numbers?

Solving Ratio and Rate Problems

Learning Goal

• using an equivalent form of a ratio or rate to solve a problem

Open Question

Materials
• calculators
• counters

Questions to Ask Before Using the Open Question

◊ *Suppose you know that the ratio of the number of boys to the number of girls in a class is 5 : 4 and that there are 27 students. How could you figure out how many are boys?* (e.g., I would put out 5 blue counters and 4 red ones but that is only 9 students. Then I would do it again and again. There are now 27 students and I can see that 3 × 5 = 15 are boys.)
◊ *Why does it make sense to say that 15 : 27 is an equivalent ratio to 5 : 9, but a more useful one for this problem?* (e.g., For this problem there are 27 students in the class and not 9, but you still always have 5 boys for every 9 girls.)
◊ *Suppose you had not used counters. What else could you have done?* (e.g., Instead of using 5 : 4, I would have used the ratio 5 : 9 for boys to the whole class. That is because I knew how many were in the whole class. I would have multiplied 5 and 9 by 3 to get an equivalent ratio where the second term was 27.)
◊ *If, instead of telling you that there were 27 students, I had to told you that there were 12 girls and asked how many boys, what would you have done?* (e.g., I would have used the ratio 5 : 4 and known it has to be ☐ : 12 and so I would have still multiplied by 3.)

Using the Open Question

Students should clearly understand that some of the things that are written on the page could be incorrect and should be verified.

By viewing or listening to student responses, note if they:
• recognize that to solve a ratio or rate problem, you use an equivalent ratio;
• are able to determine an appropriate equivalent ratio for a given one;
• are able to set up a mathematical model to match a given contextual rate or ratio situation;
• recognize the implicit information provided by a given ratio (e.g., If you know that the ratio of the number of adults to the number of children is 5 : 3, then the ratio of the number of adults to the number of all people is 5 : 8.);
• think multiplicatively rather than additively when working with ratios, (e.g., The ratio or rate $a : b$ is not equivalent to $(a + c) : (b + c)$);
• appropriately justify their reasoning when solving ratio and rate problems.

Consolidating and Reflecting on the Open Question

◊ *Could the number of children in the group for part b have been 200?* (No, e.g., The children come in groups of 3, and 200 is not a multiple of 3.)
◊ *Why does it make sense that, if you go 30 kilometres in 13 minutes, you should go a little more than 30 kilometres in 15 minutes?* (e.g., If you keep driving, more time is more distance, but it is not a lot more time, so it should not be much more distance.)
◊ *What is one way to figure out how much more distance?* (e.g., You could figure out that you go 30 ÷ 13 kilometres each minute and just add 2 of those to the 13 minute distance.)
◊ *Why did I not have to tell you the exact lengths and widths for the rectangle for you to decide about part d?* (e.g., I could just choose different examples with the correct ratio and see that they all work the same way.)

Solutions

1. e.g.,

I agree with:

b) If the ratio of the number of adults to the number of children is 5 : 3, then the ratio of the number of adults to the number of people is 5 : 8. That means that for every 5 adults, there are 8 people. Since there have to be whole numbers of adults and whole numbers of people and since 5 has no factors except for 1, there would have to be groups of 5 adults within groups of 8 people.

I disagree with:

a) If you drive 30 kilometres in 13 minutes, then you are driving 2.31 km/min. But if you drive 32 kilometres in 15 minutes, then you are driving 2.133 km/min. They are close, but not the same.

c) If you can buy \$1 U.S. with \$1.08 Canadian, you can figure how much you can buy with \$1 Canadian, by dividing 1 by 1.08. When I divided, I got almost 93¢, not 92¢.

d) If the length and width of two rectangles are in a 5 : 2 ratio, then one pair of lengths and widths is $5l$ and $5w$ and the other pair is $2l$ and $2w$. The perimeter of the first rectangle is $10l + 10w$ and the perimeter of the second rectangle is $4l + 4w$. But $10l + 10w = 5(2l + 2w)$ and $4l + 4w = 2(2l + 2w)$. The ratio is $5(2l + 2w)$ to $2(2l + 2w) = 5 : 2$.

The diagonals have length $\sqrt{25l^2 + 25w^2} = 5\sqrt{l^2 + w^2}$ and $\sqrt{4l^2 + 4w^2} = 2\sqrt{l^2 + w^2}$, and so the ratio is $5\sqrt{l^2 + w^2}$ to $2\sqrt{l^2 + w^2} = 5 : 2$.

The areas, though, do not have the ratio 5 : 2. That is because the first area is $25lw$ and the other area is $4lw$. So the ratio is $25lw$ to $4lw = 25 : 4$, not 5 : 2.

Another True Statement

If the ratio of the heights of two cylinders is 2 : 1 and they have the same size base, then the volumes are in the same ratio.

$V = \pi r^2 (2h)$ and $V = \pi r^2 h$, so the ratios are $2\pi r^2h$ to $\pi r^2h = 2 : 1$.

Another False Statement

If the ratio of the radii of two cylinders with the same height is 2 : 1, then the ratio of the volumes is also 2 : 1.

$V = \pi (2r)^2h$ and $V = \pi r^2h$, so the ratios are $\pi (2r)^2h$ to $\pi r^2h = 4 : 1$.

Think Sheet

Materials
- calculators

Questions to Ask Before Assigning the Think Sheet

◊ *Suppose you know that the ratio of the number of boys to the number of girls in a class is 5 : 4 and that there are 27 students. How could you figure out how many are boys?* (e.g., I would put out 5 blue counters and 4 red ones but that is only 9 students. Then I would do it again and again. There are now 27 students and I can see that 3 × 5 = 15 are boys.)

◊ *Why does it make sense to say that 15 : 27 is an equivalent ratio to 5 : 9, but more useful for this problem?* (e.g., For this problem there are 27 students in the class and not 9, but you still always have 5 boys for every 9 girls.)

◊ *Suppose you had not used counters. What else could you have done?* (e.g., Instead of using 5 : 4, I would have used the ratio 5 : 9 for boys to the whole class. That is because I knew how many were in the whole class. I would have multiplied 5 and 4 by 3 to get an equivalent ratio where the second term was 27.)

◊ *If instead of telling you that there were 27 students, I had to told you that there were 12 girls and asked how many boys, what would you have done?* (e.g., I would have used the ratio 5 : 4 and known it has to be □ : 12 and so I would have still multiplied by 3.)

Using the Think Sheet

Read through the introductory box with the students and make sure they understand the material explained in the instructional box.

Assign the tasks.

By viewing or listening to student responses, note if they:
- recognize that to solve a ratio or rate problem, you use an equivalent ratio;
- are able to determine an appropriate equivalent ratio for a given one;
- are able to set up a mathematical model to match a given contextual rate or ratio situation;
- think multiplicatively rather than additively when working with ratios, e.g., that the ratio or rate $a : b$ is not equivalent to $(a + c) : (b + c)$;
- appropriately justify their reasoning when solving ratio and rate problems, e.g., show different strategies to compare two rates.

Consolidating and Reflecting: Questions to Ask After Using the Think Sheet

◊ *Why does it help to think of 450 as 4 and $\frac{1}{2}$ times as much as 100 to answer Question 1b?* (e.g., You can just multiply the 10 by 4 and add 5.)

◊ *What other strategy could you have used to solve Question 2?* (e.g., I could have realized that 60 = 24 + 24 + half of 24 and then just added 32 + 32 + 16.)

◊ *What equivalent ratios were you using (or proportion did you set up) to solve Question 4?* (2 : 5 = ? : 13)

◊ *Why was finding out the price for two cans of soup useful for Question 6?* (e.g., You could do easy divisions to figure out the price for two cans and then it would be easy to see which brand's cans cost more.)

◊ *How did you solve Question 11?* (e.g., I knew that if the hexagon had side length 1, then its perimeter would be 6, so that was where I started. But if the square had side length 1, the perimeter would be 4 and the ratio would be 6 : 4. So I made the square side length $\frac{1}{2}$ and then the perimeter was 2 and I got the right ratio.)

Solutions

1. **a)** 30 L **b)** 45 L **c)** 7.5 L

2. 80 beats per minute

3. **a)** 0.333 km/min **b)** 3 minutes/km

4. 5.2 cans; e.g., I figured out that 13 is $2\frac{3}{5}$ times as much as 5. I called that 2.6 and then multiplied 2 by 2.6.

5. **a)** 454 g **b)** 1362 g **c)** 1702.5 g

6. e.g.,

   1st way: Find the cost of 12 cans of each to decide which costs less; multiply the A amount by 2 and the B amount by 3.

   2nd way: Find the cost of 2 cans of each; divide the A amount by 3 and the B amount by 2.

   3rd way: Find the cost of 1 can of each; divide the A amount by 6 and the B amount by 4.

7. **a)** 0.43 *s* **b)** 44.4 *s*

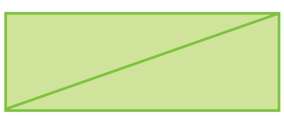
8. **a)** 339 240 babies **b)** 174 560 boys

9. 5 minutes. That is because 60 km/h means 1 km/minute. In 35 minutes, he went 35 km. But if you are going 70 km/h, since 35 is half of 70, it is half an hour or 30 minutes.

10. e.g., 30 000 cm or 300 m

11. Square sides are half as long as hexagon sides

12. e.g.,



   a) I drew a 1-centimetre line and a 3-centimetre line and "wiggled" the 3 centimetre line to be the diagonal of a rectangle.
   b) Almost 3 : 1, but not quite.
   c) Yes. I enlarged the length and width of the rectangle I had by doubling, tripling, etc., and the same thing happened.

Open Question

Solving Ratio and Rate Problems

Learning Goal

- using an equivalent form of a ratio or rate to solve a problem

Open Question

Keena's brother told her these things were true:

a) If you drive 30 kilometres in 13 minutes, then you would drive 32 kilometres in 15 minutes if you kept the same speed.

b) If there are adults and children in a large group and the ratio of the number of adults to the number of children is 5 : 3, then the total number of people has to be a multiple of 8.

c) If you buy $1 Canadian with $1.08 U.S., then you can buy $1 U.S. with 92¢ Canadian.

d) If the length and width of two different rectangles are in a 5 : 2 ratio, the ratio of their diagonals, perimeters, and areas are also in a 5 : 2 ratio.

- With which do you agree? Explain why.

- With which do you disagree? Explain why.

- Make up a similar statement that is true. Prove that it is true.

- Make up a similar statement that sounds true, but really is not true. Prove that it is not true.

Think Sheet

Think Sheet

Sometimes a situation is described using a ratio or rate, but we need an equivalent form to be able to solve a problem. For example:

Ratio problem

The Canadian flag's length-to-width dimensions are 2 : 1. We want to know how wide to make a flag that is 51 centimetres long. You want a ratio equivalent to 2 : 1 where the first term is 51.

We are trying to figure out ☐ if 2 : 1 = 51 : ☐. (× ?)

- One way to solve the problem is to figure out what we multiplied 2 by to get 51. Then we multiply 1 by the same amount. Since 51 ÷ 2 = 25.5, we multiplied 2 by 25.5 to get 51. Then the width must be 1 × 25.5 centimetres = 25.5 cm.
- Another way to solve the problem is to notice that the width is always half the length, so just take half of 51. This is also 25.5 centimetres.

Rate problem

We know that a family drove 130 kilometres in 1.6 hours. We want to know how far they would travel in 2 hours. We want an equivalent rate where the second value is 2 hours instead of 1.6 hours.

$$\frac{130}{1.6} = \frac{?}{2} \quad (\times\ ?)$$

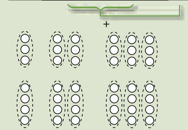
- We could figure out what to multiply 1.6 by to get 2 and then multiply 130 by the same amount.
  2 ÷ 1.6 = 1.25
  1.25 × 130 = 162.5 km
- We could also divide 130 by 1.6 (which is 81.25) to figure out the number of kilometres they drove in one hour, the **unit rate**.
  In two hours, they would go twice as far.
  2 × 81.25 = 162.50 km

- We can build a **ratio table**. A ratio table is a table where equivalent ratios or rates fill the columns. To get from one column to another, we **multiply or divide** both terms by the same amount. We can also **add or subtract** the pairs of terms in two columns to create another equivalent ratio. For example, if both ratios are equivalent to 3 : 4, they both describe groups of 3 items matching groups of 4 items; if you combine them, there are still groups of 3 items matching groups of 4 items.

× 2 × 3

3	6	9	15
4	8	12	20

+



For the problem about the family driving, we start with a rate with terms 130 and 1.6. We manipulate the values, following the rules above, to try to get a 2 in the bottom row since you want the distance for two hours. There is always more than one way to build a ratio table. One example is:

÷ 8 × 10

Distance	130	16.25	162.5
Time	1.6	0.2	2

÷ 8 × 10

1. A car uses 10 L of gasoline to go 100 kilometres. How much fuel will it use to go these distances?

a) 300 km

b) 450 km

c) 75 km

Solving Ratio and Rate Problems (Continued)

2. Tara's heart beats 32 times in 24 seconds. What is her heart rate in beats per minute? [Remember: 60 seconds = 1 minute]

3. A competitive runner goes 10 km in about 30 min.
   a) What is the unit rate in kilometres per minute?
   b) What is the unit rate in minutes per kilometre?

4. Some paint is made by mixing 2 cans of white paint for every 5 cans of red paint. To make the same tint, how many cans of white paint would you need if you used 13 cans of red paint? Explain how you solved the problem.

5. A certain recipe uses 908 g of meat for 8 servings. Figure out the amount of meat you would need for 4, 12, and 15 servings:
   a) 4 servings
   b) 12 servings
   c) 15 servings

Solving Ratio and Rate Problems (Continued)

6. 6 cans of Brand A soup cost $7.74.
   4 cans of Brand B soup cost $5.44.
   Describe three different ways to decide which brand is the better buy.

7. A computer can download a file that is 11.6 MB in one second.
   a) How long would it take to download a 5 MB file at that same speed?
   b) The same computer uploaded an 11.6 MB file in 103 seconds. How long would it take to upload a 5 MB file?

8. Canada's annual birth rate was reported as 10.28 births per 1000 people.
   a) If the population of Canada is about 33 million, about how many children are born in a year?
   b) If the ratio of male births to female births is 1.06 : 1, about how many of those new babies were boys?

Solving Ratio and Rate Problems (Continued)

9. A very cautious driver drives a certain distance in 35 minutes driving at a speed of 60 km/h. How much time would he save if he drove that distance at the legal limit of 70 km/h? Explain your thinking.

10. The scale ratio on a map is reported as 1 : 10 000. If two places are 3 cm apart on the map, how far apart are the real places? Describe your answer using two different metric units.

11. The ratio of the perimeter of a certain regular hexagon (six sides equal) to the perimeter of a certain square is 6 : 2. What do you know about the relationship between the side lengths of the two shapes?

12. a) Draw a rectangle with a width to diagonal ratio of 1 : 3. Tell how you did it.
    b) Estimate the length to width ratio.

Solving Percent Problems

Learning Goal

• using an equivalent form of a percent to solve a problem

Open Question

Materials
• calculators

Questions to Ask Before Using the Open Question

◊ *Suppose you know that 25% of the students in school take the bus. Would knowing how many students are in the school be enough information to figure out how many take the bus?* (Yes, e.g., You divide by 4.) *Why would you divide by 4?* (25% is one fourth.)
◊ *Suppose 32% took the bus. Then what would you do?* (e.g., I would divide by 32 to figure out 1% and then multiply by 100.)
◊ *In another school, I know that exactly 50 students take the bus. I also know that this is 25% of all the students. How do I figure out how many students are in the school?* (e.g., I would multiply by 4.) *Why?* (e.g., If 50 is only one fourth, you need four groups to make the whole.)
◊ *What if it were 48 students and they were only 40% of the school?* (e.g., I would realize that 10% would be 12 students and would multiply by 10 to get 100%.)

Using the Open Question

Students should clearly understand that some of the things that are written on the page could be incorrect and should be verified.

By viewing or listening to student responses, note if they:
• recognize that to solve a percent, they use an equivalent ratio;
• are able to determine an appropriate equivalent ratio for a given percent;
• are able to set up a mathematical model of some sort to match a given contextual rate or ratio situation;
• think multiplicatively rather than additively when working with ratios, e.g., that the ratio or rate $a : b$ is not equivalent to $(a + c) : (b + c)$;
• appropriately justify their reasoning when solving percent problems.

Consolidating and Reflecting on the Open Question

◊ *Why might it be faster to realize that 35% off means that you take 65% of the original price to figure out the sale cost?* (e.g., It is one less step. If you figure out how much you save, you still have to subtract that amount from the original price to get the sale price; if you use 65%, you just do one multiplication [and no subtraction].)
◊ *Why do you think it did not turn out that 10% off and then another 15% off is the same as 25% off the original price?* (e.g., When you take the 15% off the sale price, you are taking the 15% of a different amount, not the original price.)
◊ *How are the first two problems alike?* (e.g., Both times you are taking one of the percents of a different number than you are taking for the other percent.)
◊ *Why did you disagree with d?* (e.g., I tried it with a number and it did not work.) *How could you have predicted it might not work?* (e.g., You are taking 80% of one amount but 120% of a smaller one, so each percent of the big number is worth more than each percent of the small one.)

Solutions

1. e.g.,

I agree with:

c) The full price, which is 100%, is made up of the discount price (which is 35%) and the sale price. That means what is left for the sale price is 65%.

I disagree with:

a) I tried it with a $100 price. If you take 10% off, it costs $90. Then I took 15% of $90 off, which is $13.50, so the final price was $76.50. It you had taken 25% off, it would have been $75, not $76.50.

b) I tried it with a $1 price. If you take 10% off, it costs 90¢. If you pay 13% tax on $0.90, then it costs $1.02 altogether, not $1.03.

d) I know that 80 is 80% of 100. Then I had to figure out what percent of 80 the number 100 is. Since $\frac{100}{80} = \frac{5}{4}$, it is $\frac{5}{4}$ of 80 and that is the same as 125%, not 120%.

Another true statement

If you want to calculate the price of an item after 13% tax, you can just multiply by 1.13. This is true since 1.13 × a number = 1 × the number + 0.13 × the number. That is the original price plus the tax.

Another false statement that seems true

If you take 10% off a price and then put the 10% back on, it is as if you did not do anything. This is false. For example, if the price is $100, 10% off makes it cost $90. Then if you add 10% of 90 back on, you are up to $99, not $100.

Think Sheet

Materials
- calculators

Questions to Ask Before Assigning the Think Sheet

◊ *Suppose you know that 25% of the students in your school take the bus. Would knowing how many students are in the school be enough information to figure out how many take the bus?* (Yes, e.g., You divide by 4.) *Why would you divide by 4?* (25% is one fourth.)
◊ *Suppose 32% took the bus. Then what would you do?* (e.g., I would divide by 32 to figure out 1% and then multiply by 100.)
◊ *In another school, you know that exactly 50 students take the bus. You also know that this is 25% of all the students. How do you figure out how many students are in the school?* (e.g., I would multiply by 4.) *Why?* (If 50 is only one fourth, then you need four groups to make the whole.)
◊ *What if it were 48 students and they were only 40% of the school?* (e.g., I would realize that 10% would be 12 students and would multiply by 10 to get 100%.)

Using the Think Sheet

Read through the introductory box with the students and make sure they understand the material explained in the instructional box.

Assign the tasks.

By viewing or listening to student responses, note if they:
- recognize that to solve a percent, they use an equivalent ratio;
- are able to estimate percents;
- are able to determine an appropriate equivalent ratio for a given percent;
- are able to relate an equation to a percent situation;
- recognize alternative strategies for solving percent problems;
- think multiplicatively rather than additively when working with ratios, e.g., that if A is a certain percent of B, you divide and not add or subtract to determine the percent B is of A;
- appropriately justify their reasoning when solving percent problems.

Consolidating and Reflecting: Questions to Ask After Using the Think Sheet

◊ *Why does it help to think of 74% as about $\frac{3}{4}$ to solve Question 1b?* (e.g., You can just take $\frac{1}{4}$ of 80 in your head and multiply by 3.)
◊ *What was different about your questions for 2c than 2a and 2b?* (e.g., In the first parts, you knew the whole and wanted a part of it. In the last one, since you are dividing and getting a bigger answer, you know the part and not the whole.)
◊ *What strategy did you use to solve Question 6?* (e.g., I knew that 34.97 was 70%, so I divided by 7 to get 10% and then multiplied by 10 to get 100%.)
◊ *Why was your answer to Question 10b 103.2%?* (e.g., You have 100% which is the original population and another 3.2%.)
◊ *Why can you solve part B by multiplying 0.24 x 40?* (e.g., because 24% = $\frac{24}{100}$ which is 0.24 and 0.24 of something is 0.24 times it.)

Solutions

1. b and c;

   b makes sense since 74% is about $\frac{3}{4}$ and $\frac{3}{4}$ of 80 is 60.

   c makes sense since 110% of 93 is all of it and another 10%, which is close to 9. So that is 102.

2. **a)** 35% of 48 **b)** 75% of 88 **c)** 45 is 20% of a number. What is it?

3. e.g., $7. I would figure that $45.29 is close to $40. Then 10% is $4 and another 5% is another half of $4. So a tip of $6 would make sense for $40. But the meal cost was an extra $5.
   So I would add another $1.

4. **a)** I know that 10% is $8.90, almost $9. Then another 3% is about $\frac{1}{3}$ of that, so it is another $3. That means I would estimate $12.
   b) $11.57
   c) $100.57

5. e.g., because 1.13 = 1 + 0.13. The 1 refers to the original price and 0.13 is the extra tax.

6. original price: $49.96

   I realized that $34.97 was 70% of something, so I knew 1% was $34.97 ÷ 70 and the whole original price was 100 times as much.

7. **a)** about 91%
   b) 30% of $550 is $165.

8. about 127%, e.g., I used actual values to figure this out. I chose a square with a side length of 10 and another with a side length of 11.25. Then the areas were 100 and 126.56 (almost 127), so the ratio was $\frac{127}{100}$, which is 127%.

   This makes sense since the area of the large square should be more than 100% of the area of the small one, but not a whole lot more. So 127% works. Also I know that with whole numbers, areas of squares grow faster than side lengths, so it makes sense that 127 is more than 112.5.

9. about $35 900

10. **a)** 4981 people **b)** 103.2% **c)** about 96.9%

11. e.g., For A: I divide 40 by 0.24 and for B I multiply 40 by 0.24. So I am using the same numbers but doing different things with them.

Open Question

Solving Percent Problems

Learning Goal

- using an equivalent form of a percent to solve a problem

Open Question

Keena's brother told her that these things were true:

a) If you buy something on sale at 10% off and then you get another 15% off, you could have taken 25% off the original price.

b) If you buy something on sale at 10% off and then you pay 13% tax, you could have just added 3% to the original price.

c) If the discount at a store is 35%, you can calculate the sale price by using 65% of the original price.

d) If a smaller number is 80% of a larger one, then the larger one is 120% of the smaller one.

- With which do you agree? Explain why.

- With which do you disagree? Explain why.

- Make up a similar statement that is true. Prove that it is true.

- Make up a similar statement that sounds true, but really is not. Prove that it is not true.

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Think Sheet

Solving Percent Problems (Continued)

Think Sheet

Percents are ratios where the second term is 100. Sometimes, to solve a problem, we need an equivalent ratio with a different second term.

When We Know the Whole

- For example: Suppose a school has 720 students. At least 60% are required to participate in a fund-raiser before a sponsoring company will help with a school event. We want to know exactly how many students are needed. So instead of the ratio 60 : 100 (60%), we want an equivalent ratio in the form ☐ : 720.

  Solve the problem, using a fraction, decimal or ratio table:

  Using a fraction or a percent

  $60\% = \frac{3}{5}$

  So 60% of 720 = $\frac{3}{5}$ of 720 = $\frac{3}{5} \times 720$

  $\frac{3}{5} \times 720 = \frac{3 \times 720}{5} = \frac{2160}{5} = 432$

  We can use what we know about simple percents of a number (**benchmarks**) to calculate a more complicated percent.

  For example, since 10% is $\frac{1}{10}$, then 10% of 720 = 72.
  60% would be 6 times as much.
  6 × 72 = 432.

  Using a decimal

  $60\% = \frac{60}{100} = 0.60$

  0.60 × 720 = 432

  Using a Proportion

  Sometimes we solve by writing an equivalent ratio directly.

  If 60 : 100 = ☐ : 720, we divide 720 by 100 to see what we multiplied 100 by to get 72. Then we multiply 60 by that same amount.

  720 ÷ 100 = 7.2
  60 × 7.2 = 432

25 September 2011 Proportional Reasoning (I/S)

Solving Percent Problems (Continued)

Using a ratio table

We can build a **ratio table**. A ratio table is a table where equivalent ratios or rates fill the columns. To get from one column to another, we can **multiply or divide** both terms by the same amount. We can also **add or subtract the pairs of terms in two columns** to create another equivalent ratio. For example, if both ratios are equivalent to 3 : 4, they both describe groups of 3 items matching groups of 4 items; if we combine them, there are still groups of 3 items matching groups of 4 items.

We start with the ratio 60 : 100, which is 60%, and build equivalent ratios.

Step 1 It is easier to get to 720 from 20 than is from 100.

÷ 5

60	12		
100	20		720

÷ 5

Step 2 Since 720 is 700 + 20, it would be helpful to get 700 in the bottom row. It is not difficult from 100.

× 7

60	12	420	
100	20	700	720

× 7

Step 3 There is now 700 and 20 in the same row that could be combined to get 720.

+

60	12	420	432	1/5 of sixty + 7 sixties = 7 1/5 sixties
100	20	700	720	1/5 of a hundred + 7 hundreds = 7 1/5 hundreds

+

Note: We could also have multiplied column 2 by 36 (since 36 × 20 = 720) instead of getting the third column that is there. Then 12 × 36 = 432 and we would have had the same result.

26 September 2011 Proportional Reasoning (I/S)

Solving Percent Problems (Continued)

- These same strategies can be used even if the percent is more than 100% or a decimal percent; for example, if we wanted 132% of 420, we could multiply by the fraction $\frac{132}{100}$ or decimal 1.32. We could also start a ratio table with a column reading 132 and 100 or we could figure out how $\frac{132}{100} = \frac{\square}{420}$.

When We Know a Part

The same strategies we have been using can be adapted to solve problems in which we know a part and not the whole.

- For example, suppose we know that 15% of the students in a school volunteer at a local soup kitchen. We know that the number of student volunteers is 39 students. We want to know how many students are in the school. Suppose *t* is the total number of students in the school.

Using a decimal

We know that $0.15 \times t = 39$, so we divide 39 by 0.15 to figure out the missing value *t*.

$\frac{39}{0.15} = 260.$

Using a proportion

We can write the **proportion** $\frac{15}{100} = \frac{39}{\square}$. Figure out what we multiplied 15 by to get 39 by dividing 39 by 15. Multiply 100 by the same amount.

$39 \div 15 = 2.6$
$2.6 \times 100 = 260$

Using benchmarks

If we know that 15% is 39, then we know that 5% is $\frac{1}{3}$ of 39, which is 13. That means 10% is 26 (2 × 5%) and 100% is 260 (10 × 10%).

Using a ratio table

We create equivalent ratios with a first term of 39:

÷ 5 × 13

15	3	39
100	20	260

÷ 5 × 13

Solving Percent Problems (Continued)

1. Which of these make sense? Why?

a) 30% of 58 is about 12.

b) 74% of 82 is about 60.

c) 110% of 93 is about 100.

2. Each of these calculations is a step in the solution of a different percent question. Using a percent, what might the question have been?

a) 0.35×48

b) $\frac{3}{4} \times 88$

c) $45 \div 0.2$

3. People often recommend giving a 15% tip for good service. If a meal costs $45.29, what tip would you leave if you wanted to leave about 15%? Explain how to estimate without using a calculator.

Solving Percent Problems (Continued)

4. The Harmonized Sales Tax (HST) is 13%. Suppose an item costs $89 and you have to pay HST.

a) How might you estimate the HST on the item before calculating it?

b) What is the HST on that item?

c) How much is the total cost including the tax?

5. Anika calculated the price for an item including HST by multiplying by 1.13. Why does that make sense?

6. Jayda bought a sweater on sale. The discount was 30%. Before the tax was added, the sale price of the sweater was $34.97. Figure out the original price and describe the strategy you used.

7. Ethan had $550 in the bank. He took out $50 to spend on a present.

a) What percent of his money is still in the bank?

b) How much could he have taken out if he wanted to make sure that 70% was still in the bank?

Solving Percent Problems (Continued)

8. The side length of one square is 112.5% of the side length of another. What percent of the area of the small square is the area of the larger one? Explain why your answer seems reasonable.

9. A new employee in a company earns only 63% the salary of a more experienced employee. If the experienced employee earns $57 000, what would be the salary of the new employee?

10. A town of 4827 people is expected to grow by 3.2% next year.

a) What is the expected population for the next year?

b) What percent of the old population is the new one?

c) What percent of that new population is the old one?

11. What is the same and what is different about how you solve these two problems?

A: I spent $40 and it was 24% of what I had. How much did I have?

B: I spent 24% of the $40 I had. What did I spend?