GAP CLOSING

Integers

Intermediate / Senior
Facilitator's Guide

Topic 3

Integers

The Facilitator's Guide for this entire module, including the Diagnostic plus all Intervention Materials, can be downloaded at http://www.edugains.ca/resources/LearningMaterials/GapClosing/Grade9/3-Integers_FG_IS.pdf. The Student Book for this module can be downloaded at http://www.edugains.ca/resources/LearningMaterials/GapClosing/Grade9/3-Integers_SB_IS.pdf.

INTEGERS

Related Grade 9 Expectations

MPM1D

Number Sense and Algebra

- substitute into and evaluate algebraic expressions involving exponents
- simplify numerical expressions involving integers and rational numbers, with and without the use of technology
- add and subtract polynomials with up to two variables
- multiply a polynomial by a monomial involving the same variable
- expand and simplify polynomial expressions involving one variable
- solve first-degree equations, including equations with fractional coefficients
- solve problems that can be modelled with first-degree equations, …

Linear Relations

- determine values of a linear relation by using a table of values, by using the equation of the relation, …

Analytic Geometry

- express the equation of a line in the form $y = mx + b$, given the form $Ax + By + C = 0$.
- determine, through investigation, various formulas for the slope of a line segment or a line
- determine the equation of a line from information about the line

MPM1P

Number Sense and Algebra

- simplify numerical expressions involving integers and rational numbers, with and without the use of technology
- substitute into and evaluate algebraic expressions involving exponents
- add and subtract polynomials, involving the same variable up to degree three
- multiply a polynomial by a monomial involving the same variable to give results up to degree three
- solve first-degree equations with non-fractional coefficients
- substitute into algebraic equations and solve for one variable in the first degree

Linear Relations

- determine, through investigation, that the rate of change of a linear relation can be found by choosing any two points on the line that represents the relation, finding the vertical change between the points and the horizontal change between the points and writing the ratio $\frac{\text{rise}}{\text{run}}$
- determine values of a linear relation by using a table of values, by using the equation of the relation, and…

Possible reasons a student might struggle when working with integers

Students may struggle with integer operations and some even with integer representations and comparisons.

Some of the problems include:

- thinking about size of a number solely in terms of distance from zero or absolute value rather than location (e.g., thinking that –40 is more than –3 since 40 is more than 3);
- confusion between adding and subtracting, when negatives are involved;
- lack of attention to the order in which numbers are subtracted, e.g., not realizing that –3 – (–4) is not the same as –4 – (–3);
- misapplication of learned rules (e.g., applying the rule that two negatives make a positive in a situation like – 3 – 4);
- lack of fluency with whole number operations, particularly subtraction, multiplication, or division.

DIAGNOSTIC

Administer the diagnostic

Using the diagnostic results to personalize interventions

Materials
- 2-sided or 2 colours of counters
- blank number lines

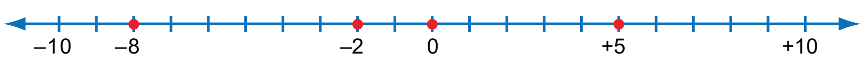
Intervention materials are included on each of these topics:
- representing and comparing integers
- adding and subtracting integers
- multiplying and dividing integers
- order of operations

You may use all or only part of these sets of materials, based on student performance with the diagnostic. If students need help in understanding the intent of a question in the diagnostic, you are encouraged to clarify that intent.

Evaluating Diagnostic Results	Suggested Intervention Materials
If students struggle with Questions 1–4	use *Representing and Comparing Integers*
If students struggle with Questions 5–8	use *Adding and Subtracting Integers*
If students struggle with Questions 9–12	use *Multiplying and Dividing Integers*
If students struggle with Questions 13–14	use *Order of Operations*

Solutions

1.



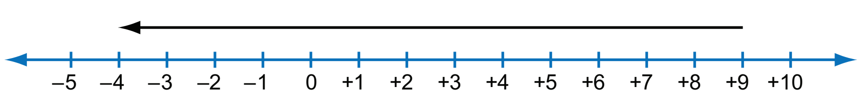
2. e.g., 2 below par on a golf course, 2° below zero for a temperature, a loss of 2 dollars

3. –20, –9, –8, –2, 3, 6, 9, +15; e.g., I know that –20 is the least, since it is negative and farthest back from 0.

4. e.g., Since 2 is farther to the right of 0 than 1, it is bigger. That means that its opposite, –2, is also farther from 0. Since it is farther to the left, it is less.

5. **a)** –11
b) –4
c) –4
d) +10

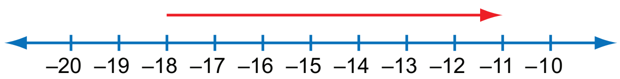
6. e.g.,



Start at 9 and go 13 back in the direction of –13 from 0.

7. **a)** 6
b) –8
c) –7
d) +7

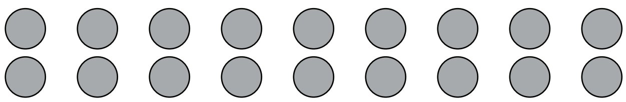
8. e.g.,



Start at –18 and count the steps and direction to get to –11.

9. **a)** –24
b) –18
c) 50
d) –63

10. e.g.,



There are 9 groups of –2.

11. **a)** 2
b) –2
c) –4
d) –4

12. e.g.,

8 negative counters are shared among 4. Each share is –2.

13. The second equation (the one with –34) is correct, e.g., *You have to do multiplication before you add, so if you multiply 8 × –4 to get –32 and add to –2, you get –34.*

14. *The second expression is 3 greater since it works out to 1, compared to the first expression that works out to –2.*

Diagnostic

1. Draw a number line from –10 to +10. Mark the locations of these integers:
–2, –8, 0, +5.

2. Describe three things that the number –2 might represent.

3. Order these integers from least to greatest: 6, –2, 3, –8, –20, +15, 9, –9.

Explain how you know which number is the least.

4. Explain why –2 < –1, even though +2 > +1.
[Recall that < means "less than" and > means "greater than."]

5. Add each pair of integers.

a) (–3) + (–8) **b)** (–20) + (+16)

c) (+9) + (–13) **d)** (+13) + (–3)

6. Use a model to show why your answer to Question 5c) makes sense.
Explain the model.

7. Subtract each pair of integers.

a) 4 – (–2) **b)** 8 – (+16)

c) (–9) – (–2) **d)** (–11) – (–18)

Diagnostic (Continued)

8. Use a model to show why your answer to Question 7d) makes sense.

9. Multiply each pair of integers.

a) (–3) × 8 **b)** 9 × (–2)

c) (–5) × (–10) **d)** (9) × (–7)

10. Use a model to show why your answer to Question 9b) makes sense.
Explain the model.

11. Divide each pair of integers.

a) (–4) ÷ (–2) **b)** (–8) ÷ 4

c) 16 ÷ (–4) **d)** (+20) ÷ (–5)

12. Use a model to show why your answer to Question 11b) makes sense.
Explain the model.

13. Circle the correct equation. Explain why it is right.

(–2) + 8 × (–4) = –24 or (–2) + 8 × (–4) = –34

14. Which of these expressions is greater? How much greater?

(–3) + 6 ÷ [4 – (–2)] or (–3) + 8 ÷ 4 – (–2)

USING INTERVENTION MATERIALS

The purpose of the suggested work is to help students build a foundation for successfully working with complex numerical expressions, including those involving exponents; operating with polynomials; and solving equations.

Each set of intervention materials includes a single-task Open Question approach and a multiple-question Think Sheet approach. These approaches both address the same learning goals, and represent different ways of engaging and interacting with learners. You could assign just one of these approaches, or sequence the Open Question approach before, or after the Think Sheet approach.

Suggestions are provided for how best to facilitate learning before, during, and after using you choice of approaches. This three-part structure consists of:
- Questions to ask before using the approach
- Using the approach
- Consolidating and reflecting on the approach

Representing and Comparing Integers

Learning Goal

- selecting a strategy to represent and compare integers depending on the values of those integers

Open Question

Questions to Ask Before Using the Open Question

◊ *Which integers are exactly 3 spaces away from 0?* (e.g., 3 and –3)
◊ *How far apart are they from each other?* (6)
◊ *Why does that make sense?* (e.g., There is a 3 on each side of 0 and 3 + 3 = 6.)
◊ *What is an integer that is less than –3?* (e.g., –4)
◊ *What makes it less?* (e.g., It is farther to the left on the number line.) *Why is it farther to the left?* (e.g., It is 4 spaces from 0 on the left and not just 3.)

Using the Open Question

Make sure students understand that there are five parts they must complete:
- choosing the eight integers as required;
- showing them on a number line;
- ordering them;
- explaining the positions of two of the negative integers; and
- describing alternate representations of two integers.

By viewing or listening to student responses, note if they:
- know how to place a negative integer on a number line;
- consider all possibilities when deciding how far apart the integers were;
- know how to order integers;
- are familiar with alternate representations or contexts for or situations involving negative integers.

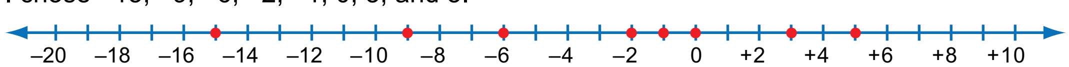
Consolidating and Reflecting on the Open Question

◊ *How did you make sure that your negative integers were never the same distance apart?* (e.g., I started at –1 and went back different distances each time. Then I checked the distances between each pair of negative integers that I used.)
◊ *How did you decide where to place your negative integers on the number line?* (e.g., I just counted back from 0 the number of spaces after the negative sign.)
◊ *Why was it easy to order the integers once they were on the number line?* (e.g., I wrote them from left to right.)
◊ *What other ways did you think of for representing negative numbers?* (e.g., I thought about temperatures and about owing money. I know you can also use two colours of counters – one for negative and one for positive.)

Solutions

e.g.,

- I chose –15, –9, –6, –2, –1, 0, 3, and 5.



- *The distances from –15 to the negatives to its right are 6, 9, 13, and 14 spaces.*
- *The distances from –9 to the negatives to its right are 3, 7, and 8 spaces.*
- *The distances from –6 to the negatives to its right are 4 and 5 spaces.*
- *The distance from –2 to the negative to its right is 1 space.*
- e.g., *I knew that –15 was 15 spaces to the left of 0. I knew that –1 was 1 space to the left of 0.*
- e.g., *–15 is a temperature that is 15° below zero. It is quite cold, but not awful.*
- *–1 is 1 below (less than) zero. You could use that if you owed someone $1.*

Think Sheet

Questions to Ask Before Assigning the Think Sheet

◊ *Which integers are exactly 3 spaces away from 0?* (e.g., 3 and –3)
◊ *What is an integer that is less than –3?* (e.g., –4)
◊ *What makes it less?* (e.g., It is farther to the left on the number line.) *Why is it farther to the left?* (e.g., It is four spaces from 0 on the left and not just three spaces.)
◊ *What are some situations where people might use negative integers?* (e.g., temperature, owing money, being below sea level.)

Using the Think Sheet

Read through the introductory box with the students or clarify any questions they might have.

If students have had experience using particular colours of counters for positive and negative instead of dark and light, have them substitute their own understanding throughout the module.

Assign the tasks.

By viewing or listening to student responses, note if they:

- know how to place a negative integer on a number line or recognize a negative integer from its placement;
- can place an integer in relation to others;
- know how to order integers;
- are familiar with situations involving negative integers;
- can generalize about the ordering of negative and positive integers.

Consolidating and Reflecting: Questions to Ask After Using the Think Sheet

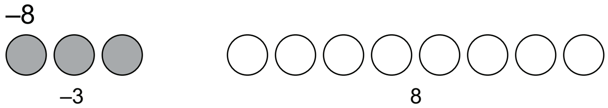
◊ *What do you think about when you are placing a negative integer on a number line?* (e.g., I look at the number after the minus sign and go left from 0 that many spaces.)
◊ *Once you are at –4 and you have to place –6, what would you do?* (e.g., I would just go back two more spaces, since I have to be two more away from 0.)
◊ *How did you decide which two opposite integers were 16 apart on the number line?* (I know that opposites are the same distance away from 0 on either side; so, I split 16 in half and realized it had to be 8 and –8.)
◊ *How did you order the integers in Questions 7 and 8?* (e.g., I know that negatives are less if the number after the minus sign is greater and that positives are less if the number is less. I know that negatives are always less than positives.)
◊ *What situations did you think of for negative integers?* (e.g., I thought about temperatures and about owing money.)
◊ *Why is any negative less than any positive?* (e.g., Negatives are always less than 0 and positives are always more.)

Solutions

1. a)

b)

2. a) 3
b) –8
c)



3. **a)** 3 and –7
b) 16 and –8
c) –13 and –3

4. e.g., 8 and –8

5. e.g., –7 could be a temperature that is 7 degrees below 0.

6. e.g., It is 0 spaces from 0 and so its opposite is also 0 spaces from 0 and that is the same spot.

7. **a)** <
b) >
c) <

8. –10, –6, –2, –1, 2, 8

9. **a)** e.g., 0 and 1
b) e.g., –7 and –8
c) e.g., –1 and –10
d) e.g., –3 and –2
e) e.g., –10 and –11

10. **a)** e.g., –4, –2, –1
b) e.g., –11, –9, –7
c) e.g., –10, 10

11. **a)** e.g., 1, 2, 3, 4
b) e.g., –11, –10, –9, –8
c) e.g., Not possible since if it is greater than –2 it is either –1, 0 or positive, but if it is less than –12, it is –13 or –14 or integers, such as that.

12. e.g., Negatives are always to the left of 0 and positives are always to the right and numbers to the right are always greater.

Open Question

Representing and Comparing Integers

Learning Goal

- selecting a strategy to represent and compare integers depending on the values of those integers

Open Question

Integers include three groups of numbers:

Positive integers: the numbers you say when you count (1, 2, 3, 4, 5, 6,…), although we often put a + sign in front of them when we are talking about them as integers (e.g., +1, +2, +3, …)

Zero: 0

Negative integers: the opposites of the counting numbers (–1, –2, –3, –4,…). Each negative integer is as far from 0 as its opposite on a number line, but on the left side of zero.

- Choose eight integers, using these rules:
  - Five of them are negative.
  - When you place the integers on the number line, no two of the negatives are the same distance apart.
- Show them on a number line and check that you have used the rules.
- Order the eight integers from least to greatest.
- Choose **two** of the negative integers. Explain how you positioned them on the number line.
- Show another way you might represent or describe those two negative integers.

Think Sheet

Representing and Comparing Integers (Continued)

Think Sheet

Representing Integers

Integers include three groups of numbers:

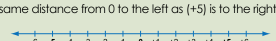
Positive integers: the numbers we say when you count (1, 2, 3, 4, 5, 6,…), although we often put a + sign in front of them when we are talking about them as integers

Zero: 0

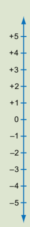
Negative integers: the opposites of the counting numbers (–1, –2, –3, –4, …)

- The most familiar use of negative integers is for temperatures. For example, –3° means 3 degrees below 0.

  Sometimes people use negative integers to describe debts; for example, if you owe $5, you could say that you have –5 dollars. Negative numbers are sometimes used in golf scores and hockey statistics.
- The negative integers are to the left of 0 on the **number line**. They are opposites of the positive integers; opposite means equally far from 0. So, for example (–5) is exactly the same distance from 0 to the left as (+5) is to the right.

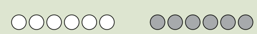


- Both positive and negative integers can be far from 0. For example, +200 is quite far from 0, but so is –200. They can also both be near to 0. For example, +1 and –1 are both very close to 0.
- We can use an up-down number line, more like a thermometer, instead of a horizontal number line, to model the integers. Here the positive numbers are above the negative ones.



Representing and Comparing Integers (Continued)

- We can also use **counters** to represent integers. We can use one colour to represent positive integers and a different colour to represent negative integers. For example, +5 could be modelled as 5 light counters and –5 could be modelled as 5 dark ones.

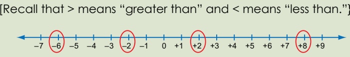


Comparing Integers

- One integer is greater than another, if it is farther to the right on a horizontal **number line** or higher on a vertical number line.

  For example, +8 > +2 and –2 > –6.

  [Recall that > means "greater than" and < means "less than."]



- It is difficult to compare integers with **counters**; number lines make the most sense to use for comparison.

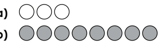
1. a) Draw a number line from –10 to +10. Mark these integers with dark dots on your number line:

+2, –6, –8, +7.

b) Mark each of their opposites with an X.

2. What integers are represented? (Remember that positive is light.)

a) 

b) 

c) Draw the opposites of each of the integers in parts a) and b) and tell what the new integers are.

Representing and Comparing Integers

3. Name two integers to meet each requirement:

a) 5 units from –2 on a number line

b) 12 units from +4 on a number line

c) 5 units from –8 on a number line

4. Two opposite integers are 16 apart on a number line. What could they be?

5. Describe something that one of the integers you named in Question 3a might represent.

6. Do you think that 0 has an opposite? If so, what is it?

7. Replace the □ with a greater than (>) or less than (<) sign to make these expressions true.

a) –2 □ +2

b) +8 □ –12

c) –12 □ –8

8. Put these integers in order from least to greatest:

–2, +8, +2, –6, –10, –1

9. List two integers to fit each description.

a) between –4 and 3

b) between –4 and –10

c) between –12 and +1

d) a little greater than –4

e) a little less than –9

Representing and Comparing Integers

(Continued)

10. Fill in the blanks so that these temperatures are in order from coldest to warmest:

a) □°, –3°, □°, □°, +1°

b) –12°, □°, –10°, □°, □°, –5°

c) □°, –5°, +2°, □°

11. List four possible values to make the statement true or explain why it is not possible.

a) an integer greater than –2 and greater than –8

b) an integer greater than –12 and less than –2

c) an integer greater than –2 and less than –12

12. Why is any negative integer less than any positive one?

Adding and Subtracting Integers

Learning Goal

• selecting a strategy to add or subtract integers depending on the values of those integers

Open Question

Materials
• integer chips (2 colours of counters)
• blank number lines

Questions to Ask Before Using the Open Question

◊ *How would you add –3 and –4?* (e.g., I would use three negative and four negative counters and that makes –7.)
◊ *Why is it easy to figure out 7 + (–7)?* (e.g., They are opposites and opposites add to 0.)
◊ *How do you know that –3 and 4 have a different sum than –4 and 3?* (e.g., One time there are more negatives and so the answer is negative; but the other time, there are more positives and so, the answer is positive.)
◊ *When you subtract two numbers, does the order matter?* (Yes, e.g., 3 – 4 = –1 but 4 – 3 = 1.)
◊ *Why is 5 – 3 less than 5 – 2?* (e.g., because you are taking more away.)
◊ *Which do you think will be less: 5 – (–3) or 5 – (–4)?* (e.g., 5 – (–3) since (–3) is greater it's like taking away more.)

Using the Open Question

Make sure students understand that they must come up with four pairs of integers three separate times – one to meet the original condition and twice more to meet each of the two created rules. Each time they must show that their choices fit the requirements.

Provide integer chips and blank numbers lines that students might use.

By viewing or listening to student responses, note:
• what models they use to add integers;
• what models they use to subtract integers;
• whether they can predict sums and so create a pair of integers with a particular sum;
• whether they recognize that if two negative integers are close, they should subtract the greater one from the lesser one to have a negative difference;
• whether they have generalized the rules for subtracting or adding integers to predict the sign of the sum or difference.

Consolidating and Reflecting on the Open Question

◊ *Why did you choose both integers to be negative?* (e.g., I wanted the sum to be negative)
◊ *Why were none of your integers less than –20?* (e.g., If one is less than –20, the other is positive and the difference is not close to zero.)
◊ *Why did you select your pairs of integers close to each other?* (e.g., I wanted to make sure that the difference between them is close to 0.)
◊ *In which order did you subtract? Why?* (e.g., I subtracted the greater one from the lesser one. That way the answer was negative.)
◊ *Could one of your rules have been a sum that is negative and a difference that is positive? Explain.* (Yes, e.g., I would have used the same numbers but subtracted in the opposite order.)
◊ *Could one of your rules have been a sum that is positive and a difference that is negative? Explain.* (Yes, e.g., I could use two positives and subtract the greater one from the lesser one.)

Solutions

e.g.

Pair 1:
–9 and –8; –9 + (–8) = –17 and that is between –4 and –20; – 9 – (–8) = –1 and that is negative and close to 0.

Pair 2:
–10 and –7; –10 + (–7) = –17 and that is between –4 and –20; –10 – (–7) = –3 and that is negative and close to 0.

Pair 3:
–4 and –6; –4 + (–6) = –10 and that is between –4 and –20; –6 – (–4) = –2 and that is negative and close to 0.

Pair 4:
–3 and –4; –3 + (–4) = –7 and that is between –4 and –20; –4 – (–3) = –1 and that is negative and close to 0.

Rule 1:
The sum of two integers is –8 or –10 and the difference is between –5 and 5.

Possible integer pairs:	–3 and –5; –3 + (–5) = –8	and	–3 – (–5) = 2
	–4 and –4; –4 + (–4) = –8	and	–4 – (–4) = 0
	–5 and –5: –5 + (–5) = –10	and	–5 – (–5) = 0
	–7 and –3: –7 + (–3) = –10	and	–7 – (–3) = –4

Rule 2:
The sum of two integers is a lot less than their difference. The sum is negative.

Possible integer pairs:	–3 and –5: –3 + (–5) = –8	and	–3 – (–5) = 2
	–20 and –21: –20 + (–21) = –41	and	–20 – (–21) = 1
	–15 and 1: –15 + 1 = –14	and	1 – (–15) = 16
	–8 and –30: –8 + (–30) = –38	and	–8 – (–30) = 22

Think Sheet

Materials
- integer chips (2 colours of counters)
- blank number lines

Questions to Ask Before Assigning the Think Sheet

◊ *What integer would you find easy to add to (–2)?* (e.g., 2, since I know that –2 + 2 is 0.)
◊ *What would –2 + (–2) be? Why?* (–4, e.g., If you have two negative counters and another two negative counters, you have 4 of them.)
◊ *How could you model 5 – 2 on a number line?* (e.g., I would start at 5 and go 2 back.)
◊ *Why could you also start at 2 and go to 5?* (e.g., You are figuring out what to add to 2 to get 5 and that is the same as subtracting.)

Using the Think Sheet

Provide two-sided or two colour counters and blank number lines. Read through the introductory box with students and respond to any questions they might have.

Assign the tasks.

By viewing or listening to student responses, note if they can:
- use a number line model to add integers, some of which are negative;
- use a counter model to add integers, some of which are negative;
- use a number line model to subtract integers, some of which are negative;
- use a counter model to subtract integers, some of which are negative;
- create pairs of integers with particular sums or differences;
- explain why $a - (-b) = a + b$;
- explain why $a - b$ is the opposite of $b - a$;
- can generalize the rules for subtracting or adding integers to predict the sign of the sum or difference.

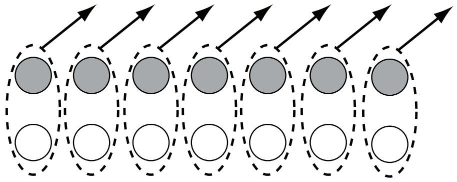
Consolidating and Reflecting: Questions to Ask After Using the Think Sheet

◊ *Why can you either add a positive and negative or two negatives to get a sum of –4?* (e.g., If you add –2 and –2, which is two negatives, you get –4. But if you add a positive to a negative that is further away from zero, you would get a negative amount.)
◊ *How did you choose two integers to have a difference of –8?* (e.g., I picked any two integers that were 8 apart on the number line and made sure to subtract the greater one from the lesser one.)
◊ *How could you choose two integers to have a sum of –8?* (e.g., You can show eight negative counters and then split them into two piles; you can add some zeroes as opposite pairs, too.)
◊ *What did you notice about the answers for Question 6f and h?* (I noticed that the numbers were the same.) *Why do you think that happened?* (e.g., 19 – (–34) is the same as 19 + 34 and that is the same as 34 – (–19), which is 34 + 19.)
◊ *How can you quickly decide whether a – b is positive or negative?* (e.g., It is positive if a is more than b and negative otherwise.)
◊ *How can you quickly decide whether a + b is positive or negative?* (e.g., It is negative if a and b are negative; it is also negative, if the distance of the negative number from 0 is greater than the distance of the positive number from 0.)

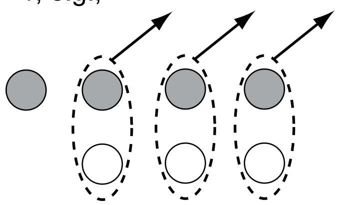
Note: It is important to avoid language such as –8 + 7 is negative since –8 is greater than 7; –8 is not greater than 7. It is, however, true that the distance of –8 from 0 is greater than the distance of 7 from 0. It is also true that 8 is greater than 7.

Solutions

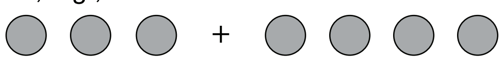
1. **a)** 0, e.g.,



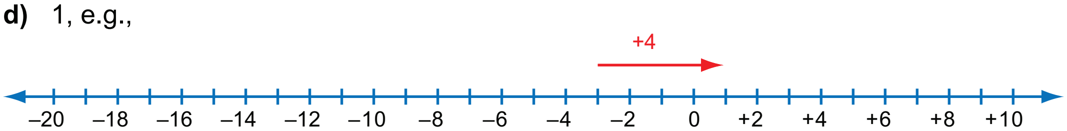
b) –1, e.g.,



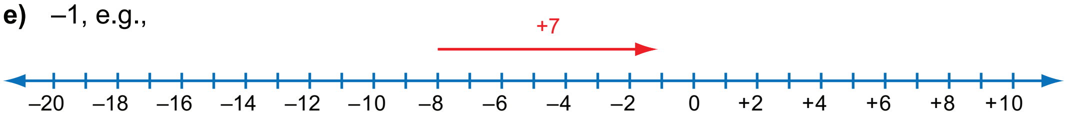
c) –7, e.g.,



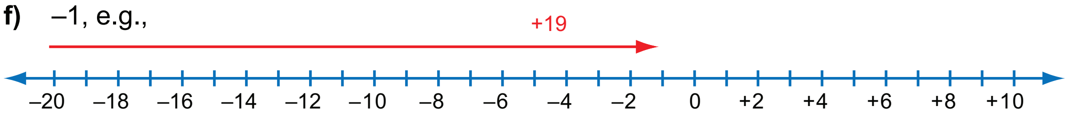
d) 1, e.g.,



e) –1, e.g.,



f) –1, e.g.,



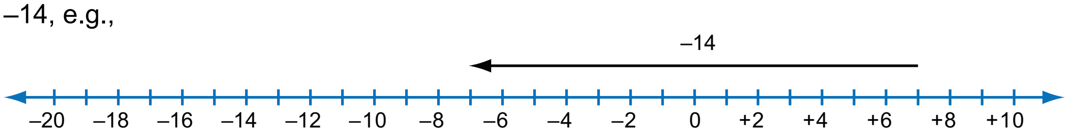
2. e.g., I could have added –4, –5, –6, and –7.
I could not have added 10, 11, 12, or 13.

3, **a)** e.g., –2 and –2
b) e.g., –10 and 6

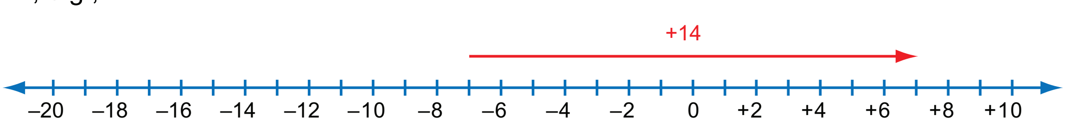
4. e.g., she added the opposites to get lots of 0s, so the sum is 0.

5. **a)** Always true; e.g., If you start at the right of zero on the number line and go farther right, you are still on the right.
b) Always true: e.g., If you start at the left of zero on the number line and go farther left, you are still on the left.
c) Sometimes true; e.g., if you add 6 and –2, the sum is positive but if you add –6 and 2, the sum is negative.

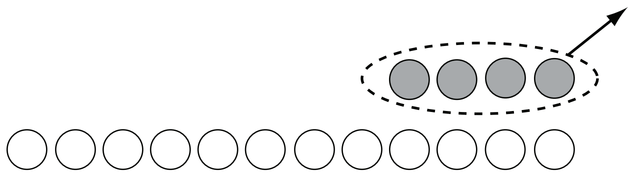
6. **a)** –14, e.g.,



b) 14, e.g.,



c) 12, e.g.,



d) –4
e) 12
f) 53
g) –15
h) 53

7. e.g., the arrow going from –4 to 5 goes one way for 5 – (–4) and the other way from –4 – 5.

8. e.g., 10 – 18

9. e.g., I could model my number by putting counters down (any counters at all). I could pretend that I did not have 7 dark ones to take away, so I add 7 dark ones and 7 light ones. Then I take away the 7 dark ones. I have 7 more light counters than I did before so it was my original number + 7.

10. a) is negative
b) is positive
c) e.g., the integer you are subtracting is closer to 0
d) the integer you are subtracting is greater

Open Question

Adding and Subtracting Integers

Learning Goal

- selecting a strategy to add or subtract integers depending on the values of those integers

Open Question

The sum of two integers is between –20 and –4. If you subtract these two integers, the difference is a negative number close to 0.

- List four possible pairs of integers.
- Explain how you know they are correct.
- Make up two other rules describing a sum and difference of two integers; either the sum or the difference or both must be negative.

Choose four possible pairs of integers to satisfy those rules. Show that they satisfy the rules you made up.

Think Sheet

Adding and Subtracting Integers (Continued)

Think Sheet

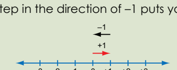
Adding

When we add two integers, we combine them, just as when we add whole numbers.

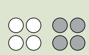
- We must also remember the **Zero Principle: –1 + 1 = 0.**

  For example: If a boy had $1 (+1) in his wallet and a debt of $1 (–1), it is as if he had $0 (or no money).

  Also moving forward one step on a number line from 0 in the direction of +1 and then moving one step in the direction of –1 puts you back at 0.

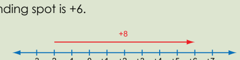


  When you have counters, the zero principle allows you to model 0 with any equal number of dark and light counters since any pair of light and dark counters is 0.



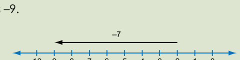
- If we add two numbers on a **number line**, we start at the first number and move the appropriate number of steps in the direction from 0 of the second number. The sum is the final location.

  For example, –2 + (+8) means: Start at –2. Move 8 steps to the right (direction of +8 from 0). The landing spot is +6.



(–2) + (+8) = (+6)

  (–2) + (–7) means: Start at –2 and move 7 steps to the left (direction of –7 from 0). The landing spot is –9.

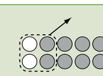


(–2) + (–7) = (–9)

Adding and Subtracting Integers (Continued)

- If we add integers using **counters**, we represent both numbers and combine them. We can ignore any pairs of counters that make 0, since 0 does not affect a sum.

  For example: +2 + (–8) is shown at the right:



+2 + (–8) = –6

Subtracting

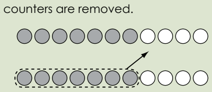
- On a **number line**, it is useful to think of subtracting by thinking of what to add to one number to get the total. For example, –3 – (–7) asks: *What do I have to add to –7 to get to –3?* The result is the distance and direction travelled.

–3 – (–7) = +4

- Using **counters**, it is useful to think of subtraction as take away.

  To model –3 – (–7), we show three dark counters, but we want to take away 7 dark counters.

  Since there are not seven dark counters to take away, we add zeroes to the three dark counters until we have enough to take away. The result is what is left after the seven dark counters are removed.



–3 – (–7) = +4

- If numbers are far from 0, we imagine the counters or number line.

  For example, on a number line (–45) – (+8) asks what to add to 8 to go to –45. We start at 8. We go 8 back to get to 0 and another 45 back to get to –45, so the result is –53.

  Or if we have 45 dark counters and want to remove eight light ones, we need to add 0 as eight pairs of dark and light counters. We remove the eight light ones and are left with 53 dark counters: (–45) + (–8) = –53.

  Some people think of *a* – *b* as *a* + (–*b*) and think of *a* + (–*b*) as *a* – *b*.

Adding and Subtracting Integers (Continued)

1. Model and solve each addition.

a) (–7) + (+7)

b) (+3) + (–4)

c) (–3) + (–4)

d) (–3) + (+4)

e) (–8) + (+7)

f) (–20) + (+19)

2. You add a number to (–3) and the sum is negative. List four possible numbers you might have added and four you could not have added.

3. You add two integers and the sum is –4.

a) List two possible negative integers.

b) List two possible integers that are not both negative.

4. Katie added all the integers from –20 to +20 in her head. Explain how could she have done that.

5. Is the statement sometimes true, always true, or never true? Explain.

a) The sum of two positive integers is positive.

b) The sum of two negative integers is negative.

c) The sum of a positive and negative integer is positive.

Adding and Subtracting Integers (Continued)

6. Solve each subtraction. Model at least three of them.

a) (–7) – (+7)

b) (+7) – (–7)

c) 8 – (–4)

d) 4 – 8

e) 4 – (–8)

f) 34 – (–19)

g) 19 – (+34)

h) 19 – (–34)

7. How can you use a number line to show why 5 – (–4) is the opposite of (–4) – 5?

8. You subtract two integers and the difference is –8. What could those integers be?

9. Use a model to explain why any number – (–7) is the same as that number + 7.

10. Complete the statement to make it true.

a) If you subtract a positive integer from a negative one, the result ____________________.

b) If you subtract a negative integer from a positive one, the result ____________________.

c) If you subtract a negative integer from a negative one, the result is negative if ____________________.

d) If you subtract a positive integer from a positive one, the result is negative if ____________________.

Multiplying and Dividing Integers

Learning Goal

- justifying the value and sign of a product or quotient of two integers

Open Question

Materials
- integer chips (2 colours of counters)
- blank number lines

Questions to Ask Before Using the Open Question

◊ *What does 3 × 2 mean?* (e.g., 3 groups of 2) *What do you think 3 × (–2) means?* (e.g., 3 groups of –2)
◊ *Why is 5 × 7 = 7 × 5?* (e.g., You can change the order when you multiply without changing the answer.)
◊ *What do you think –4 × 3 would be? Why?* (e.g., –12, since it is the same as 3 × –4 and that is three groups of –4, which is –12.)
◊ *How could you use multiplication to figure out what (–12) ÷ (–3) is?* (e.g., I could figure out what to multiply –3 by to get –12.)
◊ *Suppose you think of –12 ÷ (–3) as how many groups of –3 in –12. What result would you get?* (e.g., I would still get 4.)
◊ *Would –12 ÷ 5 be an integer?* (e.g., No, since you cannot share 12 among 5 and get a whole number)
◊ *Would 5 ÷ (–10) be an integer?* (No, e.g., It has to be –10 ÷ 5.)

Using the Open Question

Make sure students understand that they must come up with four pairs of integers three separate times – one to meet the original condition and twice more to meet each of the two created rules. Each time they must show that their choices are correct.

Provide integer chips and blank numbers lines for students to use.

By viewing or listening to student responses, note:
- what models they use to multiply integers;
- what models they use to divide integers;
- whether they can predict products and so create a pair of integers with a particular product;
- whether they effectively use what they know about factors and multiples to come up with possibilities;
- whether they realize that one integer has to be negative and one positive;
- whether they generalize the sign rules involving multiplying and dividing integers.

Consolidating and Reflecting on the Open Question

◊ *Why did you choose one positive and one negative integer?* (e.g., I did it so the product would be negative.)
◊ *I notice you chose –5 and 15. Could you have chosen 5 and –15 as well?* (e.g., Yes, since the product is the same and I could just divide the right way.)
◊ *Could one of your integers have been –1?* (No, since then the other integer would have to be 20 or greater and the quotient is not really close to 0.)
◊ *Could one of your rules have required a positive product and a negative quotient? Explain.* (No, e.g., If the product is positive, either both numbers are positive or both are negative and then the quotient would also be positive, not negative.)

Solutions

e.g.,

Pair 1: –5 and 10
–5 × 10 = –50, which is between –100 and –20

10 ÷ (–5) = –2, which is close to 0

Pair 2: –4 and 16
–4 × 16 = –64, which is between –100 and –20

16 ÷ (–4) = –4, which is close to 0

Pair 3: 6 and –12
–12 × 6 = –72, which is between –100 and –20

–12 ÷ 6 = –2, which is close to 0

Pair 4: 5 and –15
5 × –15 = –75, which is between –100 and –20

–15 ÷ 5 = –3, which is close to 0

Rule 1: The quotient of two integers is a negative integer. The product is more than –100.
Pairs of integers: –20 and 4, –18 and 3, 21 and –3; 8 and –2

Rule 2: The product of two integers is more than 50 less than the quotient.
Pairs of integers: –20 and 4, 21 and –3, –12 and 6, –14 and 7

Think Sheet

Materials
- integer chips (2 colours of counters)
- blank number lines

Questions to Ask Before Assigning the Think Sheet

◊ *How could you use what you know about integer addition to figure out 3 × (–4)?* (e.g., I would add –4 and –4 and –4 to get –12.)
◊ *What do you notice?* (e.g., I see that it is 3 × 4 with a negative sign.)
◊ *How could you use multiplication to figure out what (–12) ÷ (–3) is?* (e.g., I could figure out what to multiply –3 by to get –12.)
◊ *Suppose you think of –12 ÷ (–3) as how many groups of –3 in –12. What result would you get?* (e.g., I would still get 4.)

Using the Think Sheet

Provide two-sided or two colour counters and blank number lines.

Read through the introductory box with students and clarify any questions they might have.

Assign the tasks.

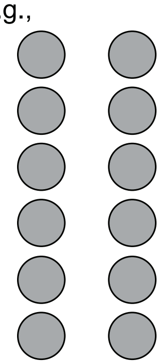
By viewing or listening to student responses, note whether they:
- use models to multiply integers and, if so, which ones;
- use models to divide integers and, if so, which ones;
- relate division of integers to multiplication;
- can create pairs of integers with particular products or quotients;
- can interpret patterns to explain sign rules involving multiplication and division of integers;
- generalize the rules for multiplying and dividing integers to predict the sign of the product or quotient.

Consolidating and Reflecting: Questions to Ask After Using the Think Sheet

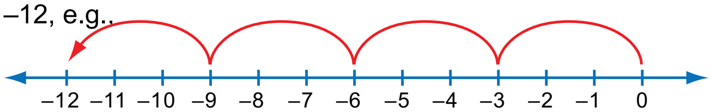
◊ *How could you explain why –4 × 9 is –36?* (e.g., I know that it is the same as 9 × –4, so I would show 9 jumps of –4 on the number line, starting at 0, and I would be at –36.)
◊ *How could you explain why (–20) ÷ 4 is –5?* (e.g., I would take 20 dark counters and share them four ways. Each share would be five dark counters.)
◊ *How could you choose two integers to have a quotient of –8?* (e.g., You can make several groups of –8 and then you divide the total amount by the number of groups.)
◊ *How can you quickly decide whether a × b is positive or negative?* (e.g., It is negative, if one is positive and one is negative.)
◊ *How can you quickly decide whether a ÷ b is positive or negative?* (e.g., It is negative, if one is positive and one is negative.)

Solutions

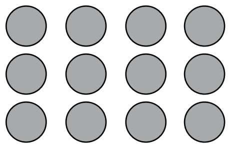
1. a) –12, e.g.,



b) –12, e.g.,



c) –12, e.g.,



d) –49
e) –56
f) –190

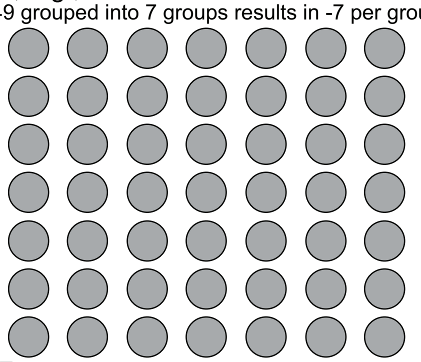
2. e.g., Possible numbers: 1, 2, 3, 4
Not possible numbers: –1, –2, –3, –4

3. a) e.g., 1 and –36, 4 and –9, –4 and 9, 6 and –6
b) –4 × (–9) is positive.
4. a) True, since you have groups of something positive, so everything is positive
b) False, since the product is always positive

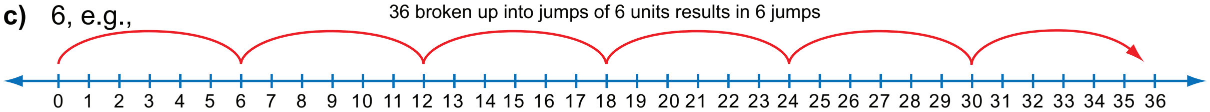
5. a) –6, –4, –2, 0, 2, 4; e.g., I notice that the numbers are going up by 2.
b) e.g., 3 × (–6) = (–18)
2 × (–6) = (–12)
1 × (–6) = (–6)
0 × (–6) = 0
(–1) × (–6) = 6
(–2) × (–6) = 12
(–3) × (–6) = 18

6. a) Yes
b) Since 3 × (2) = 6, then 3 × (–2) = –6, and since (–3) is the opposite of 3, then (–3) × (–2) = 6

7. a) –7, e.g.,
-49 grouped into 7 groups results in -7 per group



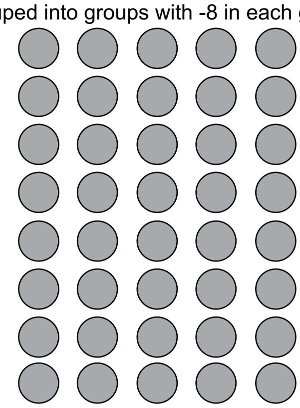
b) –7
c) 6, e.g.,
36 broken up into jumps of 6 units results in 6 jumps



d) –9
e) 11

f) 5, e.g.,

-40 grouped into groups with -8 in each group result in 5 groups



8. e.g., Since division is the opposite of multiplication, then I am looking for a number to multiply by –6 to get +30. If I multiply a negative to get a positive, the number must be negative.

9. e.g., 80 ÷ (–10), –40 ÷ 5, –80 ÷ 10

10. e.g., 12 and –1, 24 and –2, 36 and –3 and –48 and 4

11. a) 3, 2, 1, 0, –1, –2; the numbers are going down by 1

b) e.g., –9 ÷ (–3) = 3
–6 ÷ (–3) = 2
–3 ÷ (–3) = 1
0 ÷ (–3) = 0
3 ÷ (–3) = –1
6 ÷ (–3) = –2
9 ÷ (–3) = –3

12. a) is negative
b) is positive
c) is positive
d) is negative

Open Question

Multiplying and Dividing Integers

Learning Goal

- justifying the value and sign of a product or quotient of two integers

Open Question

The product of two integers is between –100 and –20. If you divide the same two integers, the quotient is an integer close to 0.

- List four possible pairs of integers. Make sure some of the divisors are positive, some are negative, and the quotients are all different.

- Show that your pairs satisfy the rules.

- Make up two other rules describing a product and quotient of two integers (some of the results must be negative).

Choose four possible pairs of integers to satisfy your rules.

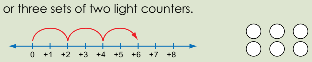
Think Sheet

Multiplying and Dividing Integers (Continued)

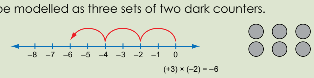
Think Sheet

Multiplying

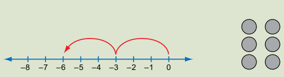
- We know how to multiply two positive integers. For example, (+3) × (+2) means three groups of +2. That is the same as three jumps of 2 starting at 0 on the number line or three sets of two light counters.



- In a similar way, (+3) × (–2) is three jumps of (–2) starting at 0 on the number line or it could be modelled as three sets of two dark counters.



- Since it does not matter in which order you multiply numbers, (–3) × (+2) is the same as (+2) × (–3). That is 2 jumps of (–3) starting from 0 or two groups of three dark counters.



  Notice that the products for (+3) × (–2) or (+2) × (–3) are the same and the opposite of the product for (+3) × (+2).

- There is no simple way to model (–3) × (–2), but it does make sense that the product should be the opposite of (+3) × (–2) and the result is +6. See Questions 5 and 6 for other ways to understand why (–*a*) × (–*b*) = +*ab*.

  Notice that (–3) × (–2) = (+3) × (+2) and (–3) × (+2) = (+3) × (–2).

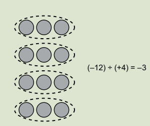
Multiplying and Dividing Integers (Continued)

Dividing

- We already know how to divide two positive integers. For example, (+12) ÷ (+4) = (+3), since division is the opposite of multiplication and (+3) × (+4) = (+12).

- In a similar way, (–12) ÷ (+4) = (–3) since (+4) × (–3) = –12.

  We model this calculation by thinking of dividing 12 dark counters into four equal groups and noticing that there are three dark counters (–3) in each group.



- (–12) ÷ (–4) = (+3) since (–4) × (+3) = (–12).

  We model this calculation by thinking: *How many groups of four dark counters are there in 12 dark counters?* Since the answer is three groups, (–12) ÷ (–4) = +3.

- It is difficult to model (+12) ÷ (–4), but it does make sense that the quotient is –3 since (–4) × (+3) = –12. See Question 11 for another way to understand why (+a) ÷ (–b) = –(a ÷ b).

  Notice that (12) ÷ (3) = (–12) ÷ (–3) and (–12) ÷ (+3) = (+12) ÷ (–3).

1. Solve each multiplication. Model parts a, b, and c.

a) (–6) × (+2)

b) (+3) × (–4)

c) (–3) × (+4)

d) (–7) × (+7)

e) (–8) × (+7)

f) (–10) × (+19)

Multiplying and Dividing Integers (Continued)

2. You multiply a number by (–3) and the product is negative. List four possible numbers you could have multiplied by –3 and four you could not have multiplied.

3. You multiply two integers and the product is (–36).
   a) List four possible pairs of integers.
   b) Explain why –4 × (–9) is not a solution.

4. Is the statement true or false? Explain.
   a) The product of two positive integers is always positive.
   b) The product of two negative integers is always negative.

5. a) Complete this pattern. What do you notice?
   3 × (–2) =
   2 × (–2) =
   1 × (–2) =
   0 × (–2) =
   (–1) × (–2) =
   (–2) × (–2) =
   b) What pattern could you create to show why (–3) × (–6) = (+18)?

Multiplying and Dividing Integers (Continued)

6. Karan says that since 3 × 4 is the opposite of –3 × 4, then –3 × (–2) should be the opposite of 3 × (–2).
   a) Do you agree with Karan?
   b) How would that help Karan figure out (–3) × (–2)?

7. Model and solve at least three of these.
   a) (–49) ÷ 7
   b) 49 ÷ (–7)
   c) 36 ÷ (6)
   d) (–81) ÷ 9
   e) (–22) ÷ (–2)
   f) (–40) ÷ (–8)

8. Why does it make sense that 30 ÷ (–6) is negative?

9. Two other integers have the same quotient as 40 ÷ (–5). List three possible pairs of integers.

Multiplying and Dividing Integers (Continued)

10. You divide two integers and the quotient is –12. List four possible pairs of integers.

11. a) Complete the pattern. What do you notice?
   (–12) ÷ (–4) =
   (–8) ÷ (–4) =
   (–4) ÷ (–4) =
   0 ÷ (–4) =
   4 ÷ (–4) =
   8 ÷ (–4) =
   b) What pattern could you create to show why (+9) ÷ (–3) = –3?

12. Complete the statement to make it true.
   a) If you divide a positive integer by a negative one, the result ________
   b) If you divide a positive integer by a positive one, the result ________
   c) If you divide a negative integer by a negative one, the result ________
   d) If you divide a negative integer by a positive one, the result ________.

Order of Operations

Learning Goal

- recognizing that the same order of operations rules that apply to whole number calculations must apply to integer calculations

Open Question

Materials
- integer chips – 2 colours of counters (optional),
- blank number lines (optional)
- blank number cards (optional)

Questions to Ask Before Using the Open Question

◊ *Suppose I asked you to add 4 to 8 and then multiply by 3. What would be the result?* (36)
◊ *Why would we not write (4 + 8 × 3)?* (e.g., That says to multiply 8 by 3 first and then add it to 4, so it is different.)
◊ *What do you remember about the rules for order of operations with whole numbers?* (e.g., I should do brackets first, then multiply and divide, and then add and subtract.)

Using the Open Question

Encourage students to use a variety of number sizes and different sets of operations in their four expressions. Suggest students might use cards and move them around to simplify the task.

By viewing or listening to student responses, note if they:
- correctly add, subtract, multiply and divide integers;
- correctly apply the rules for order of operations;
- recognize when the rules are not needed;
- show sound reasoning in creating their expressions;
- demonstrate flexibility in creating their expressions.

Consolidating and Reflecting on the Open Question

◊ *How did you make sure the results would be –5?* (e.g., I decided I would either divide –15 by 3, –10 by 2, or I would add –10 to 5 and then I figured out ways to get each of those parts.)
◊ *Did you actually need to know the order of operations rules for all of them?* (e.g., Yes, since I had to know to do brackets first or to multiply or divide first.)
◊ *Is there an expression you could have written where you did not need to know the rules?* (e.g., –5 × 5 ÷ 1 + 20; it works if you go from left to right.)

Solutions

e.g., [4 – (–2) + [(–8) ÷ (–2)]] ÷ (– 2)
(4 × 5 ÷ 2 – (–5)) ÷ (–3)
–3 + (–2) × (10 – 9 × 1)
–8 – (–6 + 3 × 4 ÷ 4)

Think Sheet

Materials
- integer chips – 2 colours of counters (optional),
- blank number lines (optional)

Questions to Ask Before Assigning the Think Sheet

◊ *Suppose I asked you to add 4 to 8 and then multiply by 3. What would be the result?* (36)
◊ *Why would we not write (4 + 8 × 3)?* (e.g., That says to multiply 8 by 3 first and then add it to 4, so it is different.)
◊ *What do you remember about the rules for order of operations with whole numbers?* (e.g., I should do brackets first, then multiply and divide, and then add and subtract)

Using the Think Sheet

Read through the introductory box with the students and respond to any questions they might have.

Assign the tasks.

By viewing or listening to student responses, note if they:
- correctly add, subtract, multiply and divide integers;
- correctly apply the rules for order of operations;
- recognize when the rules are not needed;
- problem solve to adjust expressions to result in a certain value;
- demonstrate flexibility in creating expressions that result in a certain value.

Consolidating and Reflecting: Questions to Ask After Using the Think Sheet

◊ *Did you actually need to know the order of operations rules for all of the parts of Question 1?* (e.g., Yes, since I had to know to do brackets first or to multiply or divide first.)
◊ *How did you solve Question 4b?* (e.g., I noticed that if I added 72 to 5, I would get 77, so I tried to write the first part to be 72. I knew that –12 ÷ (–2) was 6, so I tried to multiply it by 12. That worked, if I put brackets around (8 – 4)).
◊ *What expression did you create for Question 5?* (e.g., –5 × 5 ÷ 1 + 20; it works if you go from left to right.)
◊ *How did you make sure the results would be –2 in Question 6?* (e.g., I decided I would either divide –12 by 6, –10 by 5, or I would add –10 to 8 and then figured out ways to get each of those parts.)

Solutions

1. **a)** 7
b) 21
c) 25
d) –60
e) 26
f) –1

2. e.g., The brackets are in different places, so the order in which you do things changes.

3. e.g., 0 ÷ (–4) × 2 + (–6) = –6 but 0 + (–6) × 2 ÷ (–4) = 3

4. **a)** –12 ÷ (–2) × 8 – [4 × 3 + 5]
b) –12 ÷ (–2) × [8 – 4] × 3 + 5

5. **a)** e.g., 5 × 8 ÷ (–2) ÷ 4 – 3
b) All of the multiplications and divisions happened before the additions and subtractions anyway.

6. e.g., (–7 – 5) ÷ 6 and [4 – 8 ÷ (–2)] ÷ (–4)

Open Question

Order of Operations

Learning Goal

- recognizing that the same order of operations rules that apply to whole number calculations must apply to integer calculations

Open Question

- Choose values anywhere from –10 to + 10 for numbers to insert in the boxes. Some of them should be negative.
- Then choose at least three different operations to connect the boxes and add brackets if you wish. Your choice should result in a –5 when you use the order of operations rules.

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- Repeat three more times using at least some different integers.

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Think Sheet

Order of Operations (Continued)

Think Sheet

- If an expression involves more than one operation, we could interpret it different ways. There need to be rules so that everyone gets the same answer. For example, think about: (–2) – (–8) × (+2).

  If we subtract –8 from –2 first and then multiply the result by +2, the answer would be +12. But if we multiply (–8) × (+2) first to get –16 and then subtract –16 from –2, the answer would be +14.

The rules for **Order of Operations** are:

Step 1: Brackets first.

For example, for (–2) × [+3 – (–4)], subtract –4 from 3 first.

Notes: i) The brackets are not the parentheses we use for positive and negative integers, but ones that include calculations.

ii) Sometimes brackets are round ( ) and sometimes they are square [ ]; the shape does not matter. If there are brackets inside brackets, work on the inside brackets first.

Step 2: Division and multiplication calculations next, in order from left to right. It does not matter whether division or multiplication comes first.

For example, for (–12) × (–3) + (–12) ÷ (–2), first do (–12) × (–3) and then (–12) ÷ (–2) and then add.

Step 3: Addition and subtraction calculations next, in order from left to right.

It does not matter whether addition or a subtraction comes first.

For example, for (–12) – (+3) + (–4), first subtract and then add.

If we were evaluating (–15) ÷ (+3) – [(–2) × (–8) – (–4)], we would think:

(–15) ÷ (+3) – [16 – (–4)]

–5 – 20

–25.

Some people call the **Order of Operation** rules **BEDMAS**:

B stands for brackets.

E stands for exponents. (If there are squares or cubes, etc., do them before multiplying and dividing.)

DM stands for dividing and multiplying.

AS stands for adding and subtracting.

Order of Operations (Continued)

1. Calculate using the correct order of operations.
   a) (+12) + (–4) – (+3) × (–2) ÷ (–6)
   b) (–3) × (–4) + (–12) ÷ (–3) – (–5)
   c) 9 – (–8) × 3 + 24 ÷ (–3)
   d) 50 ÷ (–2 + (–3)) × (4 – (–2))
   e) [–6 + (–3) – (–4 + 8)] × 4 ÷ (–2)
   f) –6 + (–3) – (–4 + 8) × 4 ÷ (–2)

2. Why are the answers to Questions 1e) and 1f) different, even though the numbers and operations involved are the same?

3. Show that, if you start at 0 and perform the following three operations in different orders, you get different results:

Divide by –4	Multiply by 2	Add –6

4. a) Place brackets in the expression below to get a result of 31.

$-12 \div (-2) \times 8 - 4 \times 3 + 5$

b) Place brackets in the same expression to get a result of 77.

$-12 \div (-2) \times 8 - 4 \times 3 + 5$

5. a) Create an expression that would give the same result, if you calculated in order from left to right, as if you used the proper order of operation rules. Make sure to use integers and include both a division and a subtraction.

b) Explain why the rules did not matter in this case.

6. Create two of your own expressions, involving integer operations that would require knowing the order of operations rules, to get a result of –2.

