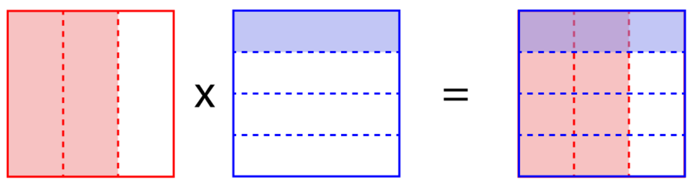
Fractions Operations: Multiplication and Division Literature Review

$$\frac{2}{3} \quad x \quad \frac{1}{4} \quad = \quad \frac{2}{12}$$



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1. Introduction

The following literature review discusses current and seminal research on fractions operations, specifically fractions multiplication and division. This review builds on a previous literature review of the foundations of fractions, *Foundations to Learning and Teaching Fractions: Addition and Subtraction Literature Review* (Bruce, Chang, Flynn & Yearley, 2013). This operations literature extends beyond the foundations review to offer new insights into the challenges of understanding fractions operations, specifically multiplication and division, and promising teaching practices that support students in a deep understanding of these procedures and their conceptual underpinnings. This document begins by outlining the methods used to conduct the literature review and then provides a comprehensive discussion of the central themes and key issues identified in the research to date on fractions multiplication and division.

Review Methods

To develop this document, a comprehensive literature review examining research on multiplication and division of fractions was completed. Relevant articles were retrieved, read and summarized. A database of reviewed articles was created and includes article citations, abstracts, brief summaries and additional notes (see appendix). Articles were selected from literature searches using the key words: "Fractions Operations," "Fractions and Multiplication," "Fractions and Division," "Fractions and Multiplications and Division," and "Multiplication and Division of Fractions" (with a focus on the latter two) in the research database ProQuest.

Summary of literature searches (as of 2013-12-19):

	ProQuest	ProQuest (Peer Reviewed)	ProQuest (Scholarly Journals)
Fractions	12190	3999	3789
Fractions Operations	790	209	190
Fractions and Multiplication	453	140	112
Fractions and Division	3289	266	247
Fractions and Multiplication and Division	294	61	55
Multiplication and Division of Fractions	285	55	49

The number of articles identified in these key word searches may appear large upon first consideration, but in fact this set is significantly smaller in size and scope compared to the total articles in consideration for the *Foundations to Learning and Teaching Fractions: Addition and Subtraction Literature Review*. Some articles were rejected as they were insufficient in their rigour of methods or in the sample size. Quantitative articles with clear and valid research methods were selected to identify trends and large-scale findings. Qualitative articles were selected to develop a fine-grained understanding of the issues of challenge and promise related to fractions operations. In total, 73 articles were thoroughly reviewed and summarized in our database, as well as 4 current and highly regarded books with sections devoted to multiplication and division with fractions.

Overview

Fractions are relational representations that can be perceived as continuous or discreet quantities, and are an integral part of our everyday lives from birth. The emphasis on whole number counting at an early age tends to reinforce a strong concept of numbers as whole numbers. When students are then, much later, introduced to non-integer number types such as fractions, they may find it difficult to transition to thinking about a continuous system or quantities that vary from ‘the whole’. “Research indicates that children have difficulty integrating fractions into their already well-established understanding of whole numbers (Staflyidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2010; Ni & Zhou, 2005), and even adults at community colleges seem to lack fundamental understanding of how to use fractions (Stigler et. al., 2010)” (DeWolf, Bassok & Holyoak, 2013, p. 389).

If foundational concepts and understanding of fractions are not addressed effectively, the groundwork is not in place for further manipulation of fractions and fractions ideas, such as considering operations contexts and procedures. Multiplication and division of fractions has proven to be a particularly difficult area to both teach and learn. This difficulty is also related to the complexity of fractions themselves as a ‘multifaceted construct’. Further, student misunderstandings of the meaning behind algorithmic ‘shortcuts’ with fractions, can lead to later problems in other areas of mathematics, such as algebra.

In this literature review, we begin by examining research on the conceptual underpinnings of multiplication and division with fractions. We then identify the current prevailing strategies for teaching multiplication and division with fractions and the related common student and teacher misconceptions. We then outline some of the more effective models and promising practices for teaching multiplication and division with fractions. In the final section we offer some recommendations for consideration and further discussion.

2. Conceptual Underpinnings of Multiplication and Division with Fractions

Multiplication and division with fractions is more complex than whole number multiplication and division (Lamon, 1999). When we consider that, in addition to the many interpretations of multiplication or division, there are also five meanings/interpretations of fractions depending on the context, we can see how complex these operations are for students. (The five subconstructs - or meanings of fractions, including part-whole, part-part, operator, quotient, and measure - are outlined in detail in the *Foundations to Learning and Teaching Fractions: Addition and Subtraction Literature Review*, Bruce et al., 2013, and discussed elsewhere in the current literature review.)

It is important to understand what is occurring when we multiply two fractions. In an area (array) model, we consider multiplication as the shared space of two numbers. In whole numbers, the shared space of 3 columns and 6 rows is 18 cells (3 x 6 = 18). With fractions, we can also think about the shared space using a partitioned area model. Important for the discussion of fraction division models below, the product of length x width (or AxB) can also be called a Cartesian product. In a fractions example, the shared space of $\frac{1}{3}$ of the area and $\frac{1}{6}$ of the area is $\frac{1}{18}$ of the whole area ($\frac{1}{3}$ x $\frac{1}{6}$ = $\frac{1}{18}$). This is illustrated in the following diagram:

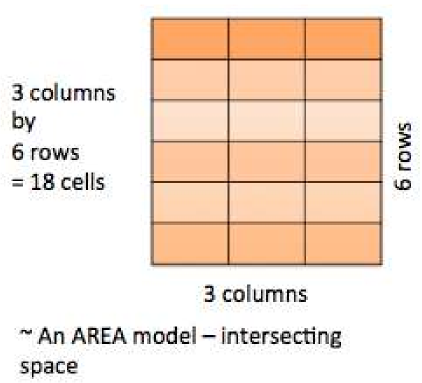


Figure 1 Area model of whole number multiplication

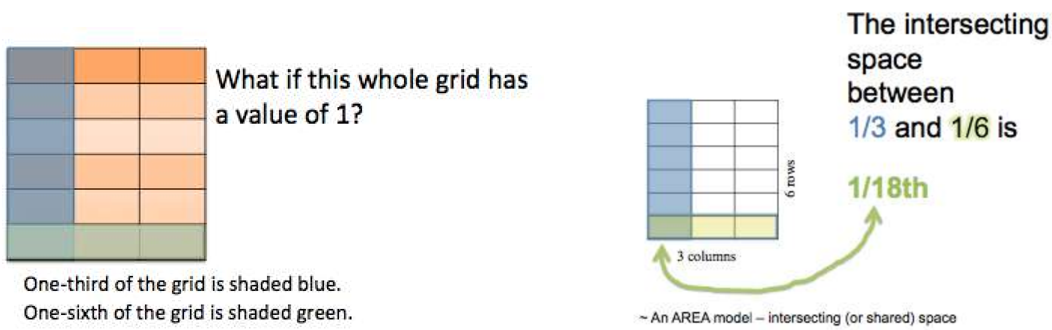


Figure 2 Area model of multiplication of fractions

In addition to the area model, whole number multiplication by skip counting can be adapted for fraction multiplication. Consider the following example of whole number skip counting (or repeated addition: 3+3+3+3+3+3 =18, or 6x3 =18:

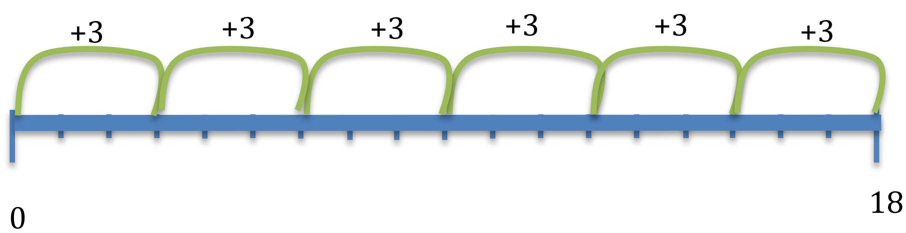


Figure 3 Number line model of whole number skip counting

For fractions, the line can run from 0 to 1 and the unit fraction can be used for repeated addition. Thinking about 18 units partitioned equally means that each jump of 3 is one sixth (the unit). In this example, we are now adding, or counting by, one-sixth units: 1 one-sixth, 2 one-sixths, 3 one-sixths, 4 one-sixths, 5 one-sixths, 6 one-sixths. This is the same as or at least similar to $6\text{x}\frac{1}{6}$.

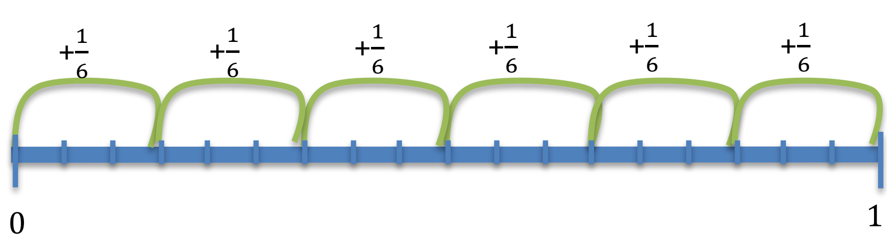


Figure 4 Number line model of skip counting fractions

The following photo is a student work example of repeated fraction addition using jumps along a number line. The students were posed the following problem: *There are 3 meters of ribbon. Each decoration needs* $\frac{2}{5}$ *of a meter of ribbon. How many decorations can you make?*

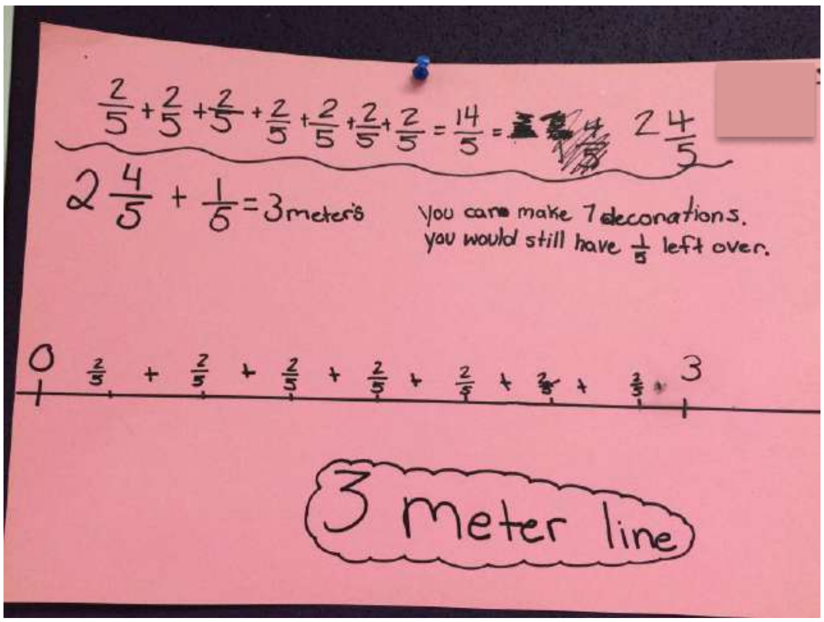
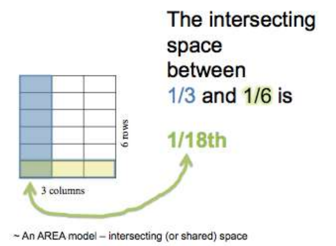


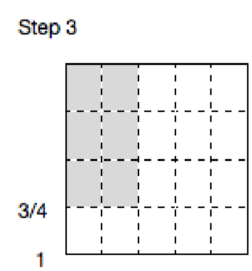
Figure 5 Sample of student solution to ribbon question

Three multiplication strategies (meaning, ways of thinking about multiplication) applied to both fractions and other number systems are outlined in the table below: (See Empson, page 189)

	Multiplication Strategy	**Description**	**Example**
Applied to other number systems	Measurement multiplication	When thinking about equal groups, the known values are usually the number of groups and the size of the groups. We use these to determine the total quantity.	A recipe calls for $\frac{3}{4}$ of a cup of flour. How much flour is needed to make $\frac{1}{2}$ of the recipe? "What is one-half of three-fourths?" $\frac{3}{4} \times \frac{1}{2} = \square$

	Multiplication Strategy	Description	Example
	"Partial groups" multiplication	In this example we are multiplying one fraction quantity with another fraction quantity.	Each bag of candy has $\frac{1}{2}$ a pound. There are $3\frac{1}{2}$ bags of candy. How much candy do I have all together?
	Cartesian product	This model considers multiplication as the shared space of two numbers.	

It is also important to understand what happens when we divide two fractions. The complexity of this operation is apparent when we consider the many interpretations for the division of fractions. There are, in fact, several different ways to think about, or models for, division, according to Yim (2010) and Sinicrope, Mick and Kolb (2002). Sinicrope, Mick and Kolb (2002) explain that we may "divide to determine how many times one quantity is contained in a given quantity, to share, to determine what the unit is, to determine the original amount, and to determine a dimension for an array" (p. 161). As with multiplication of fractions, it is helpful to relate our knowledge of whole number operations to fraction operations, and, therefore, to consider models that can be used for both division of whole numbers and division of fractions. The following table outlines division strategies as they apply to other number systems and as they relate more specifically to fractions:

	Division Strategy	Description	Example
Applied to other number systems	Measurement division (Quotative)	This model involves determining the number of groups, or how many times x goes in to y.	Consider using pattern blocks and thinking about how many blue rhombuses fit into 3 yellow hexagons – what fraction would one rhombus represent?
	Partitive division (Fair Share)	This model involves sharing something equally among friends. It involves determining the size of the group. Paper folding is a helpful way for children to understand partitive division.	If three friends share $\frac{1}{3}$ kilogram of chocolate, how much chocolate does each friend get?
	Division as the inverse of a Cartesian product (product-and-factors division)	This model is similar to the area model interpretation of multiplication described above (finding a Cartesian product). It involves determining the dimension of a rectangular area.	A rectangle has an area of $\frac{6}{20}$ square units. If one side length is $\frac{3}{4}$ units, what is the other side length? 

	Division Strategy	Description	Example
As related to fractions	Determination of a unit rate	This model emphasizes the size of one group (the unit rate).	A printer can print 20 pages in two and one-half minutes. How many pages does it print per minute?
	Inverse of multiplication	This model relies on understanding that division is the inverse of multiplication. By inverting a fraction and multiplying, the inverse is applied.	In a seventh-grade survey of lunch preferences, 48 students prefer pizza. This is one and one-half times the number of students who prefers the salad bar. How many prefer the salad bar?

(Sinicrope, Mick & Kolb, 2002)

The follow example further expands on the last strategy in the table above, "Inverse of multiplication." Since division is the inverse of multiplication, and a Cartesian product is calculated in a model for fraction multiplication (as in the area model described above), it makes sense that the inverse of a Cartesian product is calculated in a fraction division model. The following diagrams outline this inverse model:

Whole number division using the inverse of a Cartesian product model:

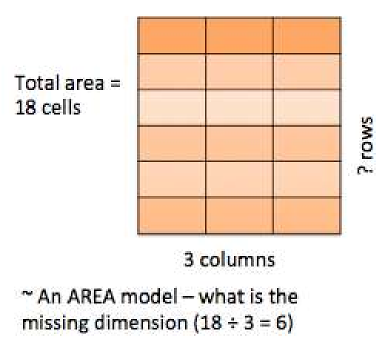


Figure 6 Area model of whole number division

Fraction division using the inverse of a Cartesian product model:

Steps to solve $\frac{6}{20} \div \frac{3}{4}$

"The area of a small rectangle is $\frac{1}{20}$, with a length of $\frac{1}{4}$ and a width of $\frac{1}{5}$. The original rectangle with an area of $\frac{6}{20}$ is composed of six small rectangles. Since the length of the original rectangle is $\frac{3}{4}$, there are three small rectangles per column. Accordingly, the original rectangle shows two columns, which means that its width is $\frac{2}{5}$."

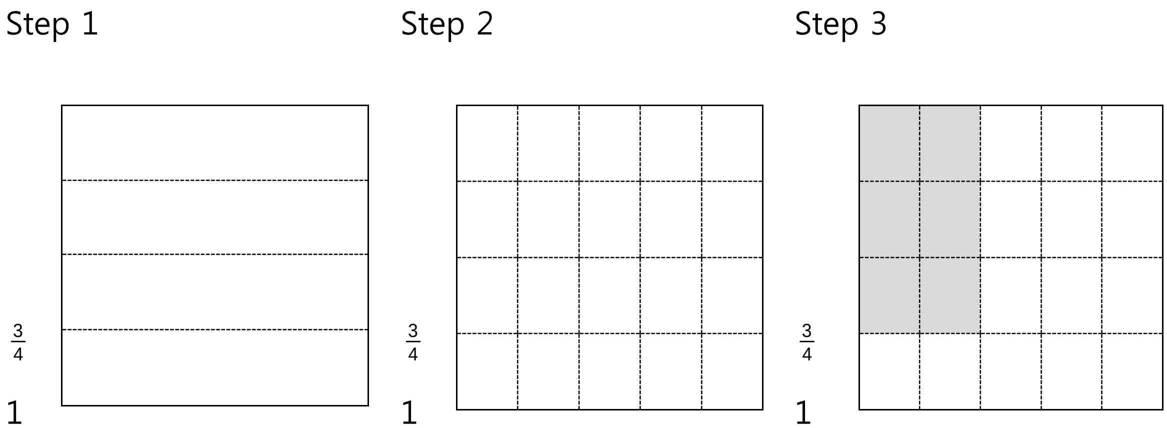


Figure 7 Area model of fraction division

(Reproduced from reference to Sinicrope, Mick & Kolb (2002) in Yim, 2010, p. 107)

In addition to the models described above, which apply to both division with whole numbers and with fractions, there are two strategies that relate specifically to the division of fractions: division as the determination of the unit rate and division as the inverse of multiplication (Sinicrope, Mick & Kolb, p. 153).

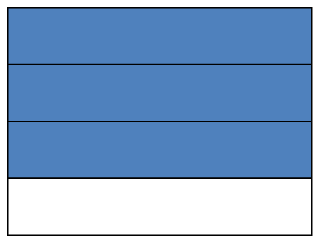
Like Sinicrope, Mick and Kolb (2002), Yim (2010) considered the first three strategies for division of fractions in a study of 10- and 11-year-olds. A review of previous work on the division of fractions indicated that there was extensive research into the measurement and partitive models of division, as described above, but not as much on the product-and-factors model ("the inverse of a Cartesian product”). The study, therefore, focused on the latter in an effort to better understand the inverse of a Cartesian product model (finding the missing dimension of a rectangular area) using pictorial procedures like the one shown above. Most of the students in the study were able to develop strategies for creating pictorial procedures. Strategies included converting either a dimension or the area to 1 (i.e. working with friendly numbers), which involves building on prior knowledge of proportional thinking about the area of a rectangle, as well as prior knowledge of multiplication and addition of fractions. Yim concluded that solving division problems in this way should prove helpful for students to better understand the meaning of fraction division algorithms (p. 119).

Lamon (1999) considered the operators that underlie fraction multiplication and division in her book on teaching fractions for understanding. She recognizes the challenges in multiplying and dividing fractions as she describes both multiplication and division of fractions as the composition of two operators (e.g., in multiplication, one combines two fractions: ‘$\frac{2}{3}$ of ($\frac{3}{4}$ of)’).

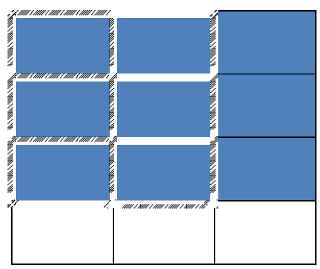
An area model is helpful in illustrating this way of thinking about multiplication. To solve this problem, a rectangle (the whole) would be divided into fourths. Three of these fourths would be shaded in (representing $\frac{3}{4}$). Then, the same rectangle (the whole) would be divided into thirds. Two thirds of the three fourths would be shaded in (representing $\frac{2}{3}$ of $\frac{3}{4}$ of 1). The overlapping area would represent the answer, in this case, $\frac{6}{12}$.

"'$\frac{2}{3}$ of' is a rule for composing the operations of multiplication and division.
'$\frac{3}{4}$ of' is a rule for composing the operations of multiplication and division.
'$\frac{2}{3}$ of ($\frac{3}{4}$ of)' is a composition of operators, defined by a composition of a composition of operations"
(Lamon, 1999, p. 101)

Step 1: Partition the whole into fourths. Shade in $\frac{3}{4}$ of it (of the whole or 1).



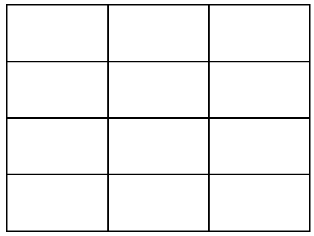
Step 2: Partition the same whole into thirds. Shade in $\frac{2}{3}$ of the $\frac{3}{4}$ (of 1).



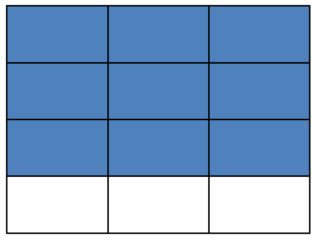
(Reproduced from Lamon, 1999, p. 101-102)

In division, one combines two fractions of 1: the question $\frac{3}{4}$ divided by $\frac{2}{3}$ can be thought about as 'how many $\frac{2}{3}$ of 1 are in there in $\frac{3}{4}$ of 1?'. Again, an area model can help with conceptualizing this. Imagine a rectangular whole. Since the fractions involved are thirds and fourths, partition the rectangle into twelfths (fourths drawn in one direction and thirds in the other). Two thirds of this area is 8 squares. Three fourths of the whole (1) is then shaded in, and two thirds of 1 is counted out, as described below:

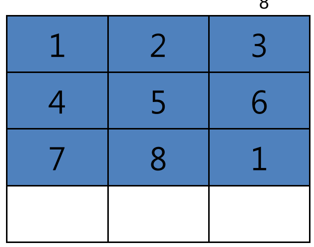
Step 1: Partition the whole into fourths (in one direction) and thirds (in the other direction)



Step 2: Shade in $\frac{3}{4}$ of the whole



Step 3: Count out how many $\frac{2}{3}$ of 1 (8 squares, since $\frac{2}{3}$ of the 12 square-whole (1) is 8 squares) fit into the $\frac{3}{4}$ shaded portion. Count the 9 squares in the shaded area by eighths: 1 to 8 (8 eighths) and then back to 1 (1 eighth), as numbered below. This shows that there are 9 eighths (8 eighths + 1 eighth) in the shaded area. Since this shaded area represents 'how many $\frac{2}{3}$ of 1 are in $\frac{3}{4}$ of 1, the answer can be written as $\frac{9}{8}$ or $1\frac{1}{8}$.



(Reproduced from Lamon, 1999: 103-104).

Informal knowledge students bring to fractions: the case of equal sharing

Equal sharing is an essential concept that underlies multiplication and division with fractions. Equal sharing involves dividing something into equal parts, for example, breaking a cookie into equal halves to share between two people.

Even very young children are able to share equally and recognize when something has been shared unfairly, or unequally. Some researchers maintain that fractions appear to be largely intuitive for very young children, who have experience and familiarity with – and curiosity about – the quantities between whole numbers (Nora Newcombe, Personal communication, August 2014).

Studies show, for example, that the ability to understand fractions as multiplicative structures stems from equal sharing problems, much like in partitive division, as discussed above (Empson, Junk, Dominguez & Turner, 2006). "Findings show that children's attempts to make sense of equal sharing elicited relationships among fractions, ratio, multiplication, and division, evidenced in how children share things exhaustively and equally among sharers" (pp. 23-24). Learning fractions often begins with equal sharing problems, where students divide up a certain amount of something among a certain number of friends (Empson, 2001). In these types of problems, students will spontaneously create examples of fraction equivalence, as they "naturally move toward the goal of partitioning or transforming shares into the biggest possible pieces" (p. 421). For example, in the equal sharing problem '24 children share 8 pancakes equally,' students were observed to divide each pancake into 24 equal parts (using the smallest pieces), but then considered how they could distribute the largest possible pieces amongst the group, (e.g., recognizing 24 as a multiple of 8 meant students partitioned pancakes into thirds) a direct application of fraction equivalence (Empson, 2001, p. 421).

Procedural vs. conceptual approaches

Often in classrooms in North America, procedural approaches to operations with fractions are emphasized over conceptual approaches (see Hasemann, 1981; this is also further discussed in Section 4 on student challenges and misconceptions). A procedural approach involves learning rules for manipulating the symbolic notation in order to produce an answer. Procedural knowledge can be defined as "'know how to do it' knowledge" (McCormick, 1997, p. 143) or as the "action sequences for solving problems." (Rittle-Johnson & Alibali, 1999, p. 175). A conceptual approach, on the other hand, provides students with the means to explore the meaning of the operation on a conceptual level, often involving exploration with hands-on materials. Conceptual knowledge can be defined as "relationships among 'items' of knowledge" (McCormick, 1997, p. 143) or as "explicit or implicit understanding of the principles that govern a domain and of the interrelations between pieces of knowledge in a domain" (Rittle-Johnson & Alibali, 1999, p. 175). Forrester and Chinnappan (2010) consider the differences between approaching a fraction multiplication problem procedurally and approaching it conceptually. A procedural approach to solving a problem would be to simply multiply the numerators together, multiply the denominators together and simplify. A conceptual approach would involve partitioning or cutting an area into a certain number of parts to find x *of* y (p. 187). Their study found that pre-service teachers mainly relied on the procedural approach, which led to errors and an inability to catch and correct mistakes. In other words, this research (along with many other studies discussed throughout this literature review) shows that an understanding of the concepts underlying multiplication and division with fractions outlined above is beneficial for deep understanding of concepts, as well as application and retention of procedures.

3. Current Teaching Strategies for Multiplication and Division with Fractions

In order to better understand the challenges students commonly face in learning to multiply and divide with fractions, we need to understand how multiplication and division with fractions are commonly presented in classrooms in North America. The literature – although slim – reveals some prevalent trends, including an emphasis on procedures over concepts; an emphasis on the part-whole meaning of fractions (combined with a lack of explicit attention to other meanings); limited use of representations of multiplication and division with fractions; as well as insufficient instructional time.

Privileging Procedures

Without conceptual understanding, students may be merely engaging in the "meaningless following of rules of calculation" (Keijzer & Terwel, 2001, p. 55). Multiplication and division with fractions typically involves all sorts of procedural and rule-based actions, such as "invert and multiply" (Rule & Hallagan, 2006). Many have noted that, while conceptualizing multiplication of fractions is difficult, the algorithms are relatively simple to memorize and apply. However, appropriate application of the algorithm does not mean a student understands the process; the traditional invert-and-multiply algorithm, for example, doesn't require an understanding of the processes behind division of fractions. (This rule also loses sight of the role of the divisor, an important piece that is often overlooked, according to Coughlin (2010); when solving word problems, for example, interpreting the solution requires reference to the divisor, since a remainder without reference to the divisor is meaningless.) Students who do not understand the reasoning behind the procedure may not always be able to apply it successfully or always know when to use it (Tsankova & Pjanic, 2009), and will eventually struggle when simply being able to follow the rule becomes insufficient (when solving more complex fraction problems, for example) (Wu, 2001).

There is evidence in the research that teachers tend to focus instruction on procedures for multiplying and dividing fractions, with less attention to the concepts underlying these procedures (how the procedures act on the fractional quantities and the reasoning behind the operations and/or algorithms themselves) (Baroody & Hume, 1991; Li, 2008; Petit, Laird & Marsden, 2010; Phillip, 2000; Rule & Hallagan, 2006). This has been attributed to at least two issues: First, research indicates that effectively teaching the underlying concepts is more challenging and requires a deeper content understanding than teaching the rules alone (Vale and Davies, 2007). And second, researchers have found that even when teachers possess solid content knowledge and conceptual understanding themselves, prospective teachers struggled to represent fractions conceptually (with pictures, diagrams or in word problems) (Lo and Luo, 2012). Thus the problem is multi-layered and requires systematic opportunities for teacher professional learning in order to support teachers in developing effective strategies and models for representing fractions concepts in ways that are meaningful and helpful to their students.

Cultural Differences in Curricula for Teaching Fractions

Analyses of curricula (in the form of textbook content) also help to document common approaches to the teaching of multiplication and division with fractions, including cultural differences. Son & Senk (2010) set out to perform a comparative analysis of curricula from South Korea and the USA. They cite a great deal of research indicating that grade 4-8 students in Asian countries, including South Korea, outperform students in economically similar European and North American countries. Textbooks from each country were analysed to assess the differences in the presentation of multiplication and division of fractions. The analysis revealed a significantly higher number of lessons and problems on the division of fractions in the Korean curriculum than the US one; there seems to be more instructional time spent on multiplication and division with fractions in Korea. Another difference is in the development of "conceptual understanding and procedural fluency" (p. 119). The researchers found that in the US curricula, instruction focuses on conceptual understanding separately from procedural fluency (e.g., the algorithms for multiplication and division), whereas in the Korean curriculum, the two are developed simultaneously. This study also found that American curricula had limited focus on the different meanings of multiplication and division with fractions ("the meaning of multiplication of fractions mainly as finding a portion of a portion and division of fractions solely using the measurement or repeated subtraction interpretation") (p. 134). In contrast, the Korean curricula uncovered and addressed multiple meanings of each operation. These observations are particularly interesting given that Korea has been reported to have higher mean achievement than the US.

4. Student Challenges and Misconceptions

Learning fractions, in general, is a serious challenge and obstacle in the "mathematical maturation of children" (Charalambous & Pitta-Pantazi, 2007: 293). Multiplication and division with fractions remains an area of struggle for students across populations in North America; for example, a study focusing on students with learning disabilities from junior-high to high school found that deficits in fraction terminology and basic fraction operations were both in the top 6 most frequently reported problem areas (McLeod & Armstrong, 1982). Although this study is now over 30 years old, current research, discussed below, concurs with the findings of McLeod and Armstrong.

Why is understanding operations with fractions so difficult?

To answer this question, we must look at the difficulties students have with fractions overall. Hasemann (1981), for example, compared the challenge of understanding operations with fractions to performing operations with whole numbers in a study of students aged 12 to 15 in Germany. Hasemann points out that fractions are more challenging because they are used far less often, and the written notation of fractions is more complex. In addition, ordering and comparing fractions (e.g., along a number line) is more difficult because we are considering multiple digits (numerator and denominator) that represent one single quantity. Further, when fractions are involved in operations, the rules and algorithms are more complicated.

Further to these complexities, fractions also possess several meanings (known in the research as "subconstructs") depending on their context and use, namely part-whole, part-part, operator, quotient, and measure (see Bruce et al., 2013, for more detailed information). Often the different meanings of fractions are not made explicit to students; in general, instruction in North American classrooms dwells particularly on the part-whole construct (Moseley

& Okamoto, 2008; Moseley, 2005). Charalambous & Pitta-Pantazi (2007) looked at differences in over 600 fifth- and sixth-grade student understanding of each of the fractions subconstructs and how this understanding affected student performance on operations with fractions as well as determining fractional equivalents. Their findings show that students are most proficient in part-whole subconstruct tasks and least proficient in measure-related tasks such as comparing fractions, ordering fractions, placing fractions on number lines and finding equivalent fractions using number lines (p. 302-304). This is not surprising given the typically strong focus on the part-whole construct in fraction teaching.

Student development of fractions subconstructs is a complex phenomenon. In a qualitative study of sixth-grade students, Hackenburg and Tillema (2009) recognize the changes in difficulty level of fraction multiplication questions; difficulty depends on the type of fractions being multiplied (e.g., a unit fraction vs. a proper fraction, and in which order they are multiplied – calculating a unit fraction of a proper fraction or a proper fraction of a unit fraction) (p. 16).

In their chapter, "Understanding Operations on Fractions and Decimals," Empson and Levi (2011) discuss the particular conceptual challenges students face when learning multiplication and division of fractions. Multiplication and division involve unfamiliar concepts when numbers involved in the question are fractions (especially when a fraction is being multiplied by a fraction, instead of a fraction by a whole number). Empson and Levi (2011) call these types of questions "partial groups problems", as the number of groups is a fraction, or partial number, and not a whole number and, therefore, involves different procedures than those students are familiar with in whole number multiplication and division. When multiplying and dividing with partial numbers, students must develop a conceptual understanding of new algorithms. As Empson and Levi (2011) explain, "These problems pose new conceptual challenges, because they involve working with parts of parts and relating a part to two different units" (p. 189). One example these authors provide is the following problem: I have $\frac{3}{4}$ of a bag of candy. A full bag of candy weighs $\frac{1}{2}$ pound. How many pounds of candy do I have? To solve this problem successfully, students need to cognitively consider $\frac{3}{4}$ of $\frac{1}{2}$ of 1 pound and the same amount of candy is both $\frac{3}{4}$ of a bag and $\frac{3}{8}$ of a pound" (p. 189).

Like Lamon (1999) and Empson & Levi (2011), Graeber and Tirosh (1990) consider the conceptual misunderstandings and overgeneralizations that students hold around multiplication and division. In the case of decimals for example, which are deeply connected to fractions as another system for non-whole numbers, this international study (involving the United States and Israel) revealed the assumptions that fourth and fifth graders brought to multiplication and division with decimals. They identified the student notion that multiplication always results in a bigger number and division in a smaller number, as a problematic belief that impeded an accurate understanding of decimals, and in turn, of fractions operations (Graeber & Tirosh, 1990).

Related to this misunderstanding is the challenge for students of managing the nuanced meanings of mathematical notation, when some familiar conventions with operations involving whole numbers suddenly mean something different. For example, when multiplying whole numbers, the "x" symbol is often interpreted as repeated addition, but when multiplying fractions, it represents taking one amount "of" another (e.g., $\frac{1}{6}$ of $\frac{1}{2}$). Student learning and flexibility with these conventions presents a significant challenge (Ott, 1990).

Longer term implications of student challenges with multiplication and division with fractions

Many researchers have noted the widespread difficulty that students have in attaining the concepts involved in multiplication and division with fractions (Brown & Quinn, 2007; Empson & Levi, 2011), and emphasize that the lack of conceptual understanding in this area has wide and serious implications that extend to other areas of mathematics (Baroody & Hume, 1991).

One of the obvious long-term implications of student memorization of algorithms without understanding is the potential of large and unreasonable errors that go unchecked. Hasemann (1981) describes these computational errors as “nonsensical results” (p. 81) attributable to a lack of reasoning or meaning behind the algorithm. When students are able to understand the underlying concepts, on the other hand, they are able to better handle increasing complexity and to apply the reasoning behind the algorithm flexibly and with greater accuracy (Baroody & Hume, 1991; Petit, Laird & Marsden, 2010).

A second long-term implication is revealed by Brown and Quinn (2007), who discuss the deep connections between fractions multiplication and division, and algebra. Their study compared 191 students in the areas of algebra and fractions competency. They found that student understanding of fraction knowledge was clearly linked (statistically significant) to algebraic reasoning. “Elementary algebra is built on a foundation of fundamental arithmetic concepts” (p. 8). In other words, algebraic concepts are similar to and rely on fractions concepts. The rational number system, for example, is introduced in early algebra and draws upon an understanding of the common fraction (p. 8). When ‘shortcuts’ such as to ‘simply cross multiply’ when dividing fractions are misunderstood or simply not connected to the reasoning of the actions, students can have difficulty with more complex algebra later on. As Brown and Quinn state, "the list of algebraic generalisations that rely on fractional constructs grows as students move to each subsequent level of mathematics" (p. 8). These generalizations include combining like terms (either of a unit fraction or of a variable) and multiplying by a constant to simplify (to either clear a fraction's denominator or clear a variable). The implications of experiencing problems in algebra are serious, as algebra has been identified as a key precursor to later mathematics learning: "If algebra is for everyone, then all students must first become familiar and fluent with fractions" (Brown & Quinn, 2007, p. 12). Gaining a solid understanding of the connected concepts and procedures related to multiplication and division with fractions is, therefore, extremely important to later success in mathematics.

5. Teaching Challenges

There is a significant amount of content knowledge required in the successful teaching of multiplication and division with fractions. Teachers themselves must not only understand *how* to multiply and divide fractions (procedural understanding), but they must have a sufficient conceptual understanding of operations in order to successfully teach the concepts. Izsák (2008) emphasizes the necessity of in-depth understanding of the content in order to be able to respond flexibly and effectively to the great range of student responses. Students benefit when teachers are able to adapt to a variety of student needs: "teachers need to reason explicitly and flexibly with nested levels of units if they are to respond to the variety of quantitative structures… that their students might assemble" (p. 104). Izsák's study highlights the importance of "explicit, flexible attention" (p. 139) to the range of student responses and what these responses imply or foreshadow in terms of student understanding or lack thereof. Given the systemic underdevelopment of fractions understanding in students and adults in North America, it is not surprising that teachers face tremendous challenges in responding flexibly and effectively to the range of student needs.

Pre-service Teaching

General Challenges

Most of the research on teaching challenges related to multiplication and division with fractions reports on issues that pre-service teachers face, and that these issues are not unlike those of younger students. Graeber, Tirosh & Glover (1989) found that pre-service teachers did, in fact, follow "primitive" models of multiplication and division and experienced difficulties with selecting the appropriate operation when solving word problems. A much more recent study of pre-service elementary teachers' procedural and conceptual knowledge of fractions in Flanders also found that the misconceptions and understandings of teachers mirror those of elementary school students (Van Steenbrugge, Lesage, Valcke & Desoete, 2014). In general, pre-service teachers' knowledge of fractions was limited: teachers in both the first year and third year of their program made many errors when tested on procedural and conceptual knowledge of fractions in this study. The authors argue that their work "provides ground to address teachers' preparation as an effective way to increase standards expected of students [student teachers]" (p. 156).

Lack of Understanding of a Fraction as a Number

Another particular similarity between student and pre-service teacher gaps in understanding appears to rest with understanding fractions as quantities. In their study of pre-service teachers, Park, Güçler & McCrory (2013) review literature on K-8 students, and note that both K-8 students and pre-service teachers have difficulty "conceiving of fractions as numbers as an extension of whole numbers…" (p. 458). In the analysis of lessons and resulting student concept attainment, the study showed that, "key mathematical aspects of fraction, including fraction-as-number, were not explicitly addressed" (p. 475). It was assumed that fraction-as-number was already understood by pre-service teachers, and was therefore not addressed in the mathematics courses. The authors argue the importance of being "aware that understanding fractions as numbers is not trivial either to mathematicians in the past or to today's K-8 students" (p. 477).

Lack of Concept Understanding

Many studies that examine pre-service teachers' mathematical content knowledge demonstrate that this knowledge may be insufficient for effective teaching of multiplication and division with fractions (Izsák, 2008; Lubinski, Fox & Thomason, 1998; Olanoff, 2011; Ball, 1990; Tobias, Olanoff and Lo, 2012). In their review of fractions research, Tobias, Olanoff and Lo (2012) found that pre-service teachers had an underdeveloped proficiency in fractions, and in later algebra (p. 668). In particular, the studies reviewed by Tobias, Olanoff and Lo (2012) demonstrate that pre-service teachers tend to: "have a rule-based conception of fraction multiplication and division" (p. 671); and have "misconceptions [that] result from overgeneralized rules from other number systems, such as multiplication always makes bigger, or result from not understanding algorithms for multiplying and dividing fractions" (p. 671). In their study of Taiwanese pre-service teachers' knowledge of fractions, Huang, Liu & Lin (2008) also found that these aspiring teachers showed better understanding of fractions procedures than fractions concepts, and had the most difficulty with operations (multiplication and division). Similarly, Forrester and Chinnappan (2010) found that pre-service teachers in Australia mainly relied on procedures rather than conceptual understanding, which led to errors of their own, as well as an inability to catch and correct student errors.

Challenges of Division with Fractions

Newton (2008) found that student teachers "were most uncertain about dividing fractions, followed by subtracting, multiplying, and adding fractions" (p. 1100). Other researchers have attempted to delve into this phenomenon to investigate further the particular difficulty with division with fractions. Ball (1990), for example, interviewed prospective teachers to assess their understanding of division by zero (e.g., $7 \div 0 = ?$) and division of fractions in algebraic equations. Many of the pre-service teachers were able to arrive at correct answers, but few could explain the foundational principals and meanings of division. Qualitative data in the study highlighted gaps in content knowledge; one teacher candidate "seemed to get stuck by his knowledge of the algorithm 'invert and multiply'" (p. 136); another explained "that she hadn't done this since high school" (p. 136). Ball conjectured that "The prospective teachers' knowledge of division seemed founded more on memorization than on conceptual understanding. Some of the teacher candidates could not remember the rules at all… Sheer memorization serves well to display mathematical knowledge in school – until one forgets, that is" (p. 141-142).

To identify the precise issues pre-service teachers were having with the division operation of fractions, Isik & Kar (2012) conducted a case study analysis of error type among pre-service elementary mathematics teachers in Turkey. Seven types of errors were identified: unit confusion; assigning natural number interpretations to fractions; problems using ratio proportions; being unable to establish part-whole relationships; dividing by the denominator of the divisor; using multiplication instead of division; and increasing errors by inverting and multiplying the divisor fraction (p. 2-7). Further, the pre-service teachers in the study lacked sufficient understanding to pose division of fractions problems that would benefit student learning, and may even have further contributed to student challenges. Given the abundance of research on pre-service teacher understanding of fractions, it is highly likely that increased attention on fractions understanding in teacher education programs would benefit these aspiring teachers and their future students (Luo, Lo & Leu, 2011; Lin et al., 2013).

In-service Teaching

Research on in-service teachers was far less abundant compared to that of pre-service teachers and suggests that there is a need for additional studies that identify the role and impact of in-service-level professional learning programs on teacher content understanding and related teaching practices. One small but hopeful study (Flores, Turner & Bachmann, 2005) of two in-service teachers focused on building teacher conceptual understanding of division with fractions and resulted in the development of a very precise sequence of types of fractions division questions to use with students.

Although it is unclear as to whether this sequence has been field-tested more widely, it is worthy of consideration:

1. Use fractions with the same denominator, so that the first is bigger than the second, and the quotient is a whole number; for example, $\frac{4}{6} \div \frac{2}{6}$.
2. With the same conditions, but the quotient does not have to be a whole number: $\frac{5}{6} \div \frac{2}{6}$.
3. Use fractions that do not have the same denominator but that are well known, are related, and have common factors, such as $\frac{1}{2} \div \frac{1}{4}$, and $\frac{1}{6} \div \frac{1}{3}$. With these examples, we want the children to find, by using manipulatives, the common denominator. What smaller piece will fit into the two fractions?
4. Now with more difficult examples, as with $\frac{2}{3} \div \frac{2}{4}$. They also need to be able to find the common denominator.
5. Ask students how we obtain the answer, look at the numbers, and search for patterns and similarities:
$$\frac{2x4}{6x4} = \frac{8}{12}$$
They are multiples/factors.
6. Look at the original problem $\frac{2}{3} \div \frac{1}{4}$ (multiplying by 4)… Explain why the algorithm works."

(Reproduced from Flores, Turner & Bachmann, 2005, p. 199)

6. Effective Strategies for Teaching Multiplication and Division with Fractions

Although there is a limited amount of published research on teacher knowledge of multiplication and division of fractions, there is a more robust literature on best practices. Effective strategies for teaching described in the research literature clustered around six key areas, as follows:

1) Increase the focus on conceptual understanding;
2) Recognize and draw on students' informal knowledge and prior experiences;
3) Draw on student familiarity with whole number operations;
4) Include multiple representations to convey meaning;
5) Specific suggestions for improving understanding of multiplication with fractions;
6) Specific suggestions for improving understanding of division with fractions;

Each of these themes is elaborated upon in the discussion below.

Effective teaching of fractions operations includes an increased focus on conceptual understanding

Research shows that student learning in multiplication and division with fractions benefits from attention to conceptual understanding before or in conjunction with algorithms (Baroody and Hume, 1991; Li, 2008; Petit, Laird & Marsden, 2010; Phillip, 2000; Rule & Hallagan, 2006): "Students need to develop number and operation sense before learning how to apply these terms through procedures, understanding what the problem means, rather than merely computing an answer" (Rule & Hallagan, 2006, p. 3). Li (2008) re-iterates that it is not enough to teach only the invert-and-multiply algorithm when teaching division of fractions; it is necessary for students to understand concepts beyond memorizing the rote calculations. In Li's Chinese textbook example, students are introduced to the meaning of fraction division (e.g., "as the inverse operation of fraction multiplication through a discussion of three related word problems") before being taught the algorithm (p. 549).

Baroody and Hume (1991) offer a developmental perspective to instruction that draws on student strengths and prior knowledge and focuses on meaning and understanding. They report that instruction should focus on: understanding, informal knowledge, purposeful learning, reflection and discussion (pp. 55-56). When learning fractions operations, Baroody and Hume (1991) suggest starting with context problems, so students can focus on what they are solving instead of how they are solving. When solving problems, it is also important for students to use appropriate manipulatives, such as pattern blocks, Cuisinaire rods and paper folding – hands on experiences which help to develop conceptual understanding. (See Rule & Hallagan, 2006, for example.)

Based on the authors' previous *Foundations to Learning and Teaching Fractions: Addition and Subtraction Literature Review* (see Bruce et al. 2013), an increased focus on understanding should include explicit teaching of the different meanings of fractions beyond part-whole relationships (to include the part-part, quotient, linear measurement and operator meanings of fractions). The findings of Hasemann (1981) and Charalambous & Pitta-Pantazi (2007) concur and "suggest that a profound understanding of the different interpretations of fractions can uplift students' performance on tasks related to the operations of fractions and to fraction equivalence" (Charalambous & Pitta-Pantazi 2007, p. 311). In other words, Charalambous and Pitta-Pantazi are recommending that a solid foundation in developing understanding of the multiple meanings of fractions (as listed above) enables and fosters understanding of what it means to operate on and manipulate fractions.

Effective teaching of multiplication and division with fractions recognizes and draws on students' informal knowledge with fractions as well as prior knowledge and experiences

Informal knowledge is prior, "real-life circumstantial" knowledge that a student can draw upon when solving problems (Mack, 1990, p. 16). Studies show that students do indeed have informal knowledge of fractions, but that they lack an understanding of algorithmic procedures and fraction symbols (p. 29). "The results add more evidence to the argument in favor of teaching concepts prior to procedure" (Mack, 1990, p. 30). Mack (2001) considered ways to build on fifth grade students' informal knowledge of multiplication with fractions. Partitioning and unit fractions are two examples of prior knowledge identified in the study. Students benefit from having a flexible understanding of the unit and determining what constitutes the whole when learning to multiply fractions and thinking about finding "a part of a part of a whole" (p. 269). The fifth graders in Mack's study (2001) also continuously drew upon their knowledge of partitioning when they worked on solving increasingly complex multiplication problems. Students reconceptualized and partitioned units to reflect the different multiplication problems they were given and adjusted their strategies based on the "relationship they perceived between the denominator of the multiplier and the numerator of the multiplicand" (p. 292). Naiser, Wright & Capraro (2004) report on their study of activating student prior knowledge, reviewing and practicing problems, and making real-world connections (p. 195) to increase student engagement and motivation. The student gains were attributed to the use of manipulatives and efforts to facilitate student construction of their own content knowledge.

According to Flores (2002), making connections amongst mathematics ideas and understanding multiple meanings of fractions are key to developing a profound understanding of division with fractions. Flores concludes that educators can help students by connecting division with fractions to other mathematical concepts, like ratio, reciprocals, inverse operations, multiplication, proportional reasoning and algebra. Inevitably, students approach division of fractions with previous knowledge of division, and it is important to optimize this prior knowledge. Equivalent fractions, multiplication and division of whole numbers, and multiplication of fractions are identified, by Flores, as important areas to draw upon.

Effective teaching of fractions multiplication and division should build from student familiarity with whole number operations

Petit, Laird and Marsden (2010) note that connections to prior knowledge about whole number operations can be powerful for students during instruction on multiplication and division with fractions. While operations with fractions certainly carry "new and different interpretations" compared with operations with whole numbers (Wu, 2001, p. 174), many processes that apply to whole numbers do still apply to fractions. The examples given by Petit, Laird and Marsden (2010) include the fact that one times a number will equal that number; 0 times a number will equal zero; and that multiplication and division are the inverse of each other. Specifically, students should "interact with a variety of situations and contexts that include both partitive and quotative division, and different kinds of remainder" (p. 178).

Multiplication

Being able to see multiplication with fractions as an extension of whole number multiplication is important to student success (Wu, 2001). In their case study, Vale and Davies (2007) discuss the connection between multiplication, division and developing an understanding of fractions, proportions and ratio. They highlight multiplicative thinking as a necessary foundation to solving fraction problems. Setting up array formations and grids for whole number multiplication and division are directly linked to using area models and grids for calculating fraction multiplication and division. The area model of multiplication provides a visual and allows connections to be made between whole number multiplication and fraction multiplication, and also helps to explain why multiplication of fractions results in a smaller number (Wu, 2001). The common approach to multiplication draws upon repeated-addition, a well-known strategy for whole number multiplication that is more difficult to apply to fraction multiplication (e.g., when thinking about $\frac{3}{4}$ x $\frac{1}{4}$, it is hard to imagine adding $\frac{3}{4}$ to itself $\frac{1}{4}$ times).

Tsankova and Pjanic (2009) also discuss the area model of multiplication as an effective way of linking multiplication with fractions to whole number multiplication: "the concept of multiplication applied in finding the area of a rectangle connects with the prior understanding that students have about multiplying natural numbers" (p. 284). Considering the overlapping parts of an area model as well as folding paper and using number lines are helpful strategies for multiplying fractions (p. 282-283).

Division

Although division with fractions is significantly different than division with whole numbers (see Section 2: Conceptual Underpinnings for Multiplication and Division with Fractions), according to Kribs-Zaleta (2008), building on knowledge of division of whole numbers can also help build understanding of division with fractions. In fact, Sidney and Alibali (2012) found that relating the abstract division structure used in whole number division was more beneficial to student learning of division with fractions than was relating knowledge of other fraction operations. The measurement model (repeated subtraction), which allows you to make as many groups as possible when the size of the group is known, and partitive division (fair sharing), which helps you divide items among a known number of groups are two such examples of whole number division models. Kribs-Zaleta (2008) worked with a group of sixth graders and found that cutting up oranges and using containers of lemonade proved helpful in illustrating models for division of fractions. The following explains the solution process seen in the study:

Consider the following problem when working through the following steps for measurement division: There are 3 meters of ribbon. Each decoration needs $\frac{2}{5}$ of a meter of ribbon. How many decorations can you make?

> For measurement division problems… most solutions involved a two-step process:
>
> 1. Subdivide the dividend into units of the given denominator (e.g.,fifths).
> 2. Group the new pieces according to the numerator (e.g., two).
>
> Solutions to the partitive problems also tended to involve two steps, but here the order was reversed…
>
> 1. Partition the dividend into as many groups as the numerator (e.g., two.
> 2. Build as many of the groups created above as the denominator (e.g., five). (Kribs-Zaleta, 2008, p. 455)

The following sample of student thinking demonstrates the strategy outlined above for measurement division problems. The student has constructed the three meters of ribbon, partitioned each meter into fifths and then grouped into two-fifths to determine the number of groups.

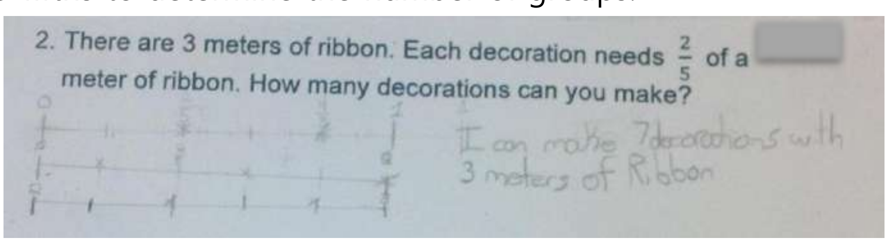


Figure 8 Student solution demonstrating measurement division strategy

An effective way of helping students understand the invert-and-multiply algorithm is to relate it directly to commonly taught whole number division interpretations: sharing and measurement. The following table is taken from Siebert (2002) and outlines how each of these models can explain why inverting the second fraction and multiplying makes sense for fraction division.

Summary of the measurement and sharing interpretations for division of fractions

	Measurement	**Sharing**
Situations	Joel is walking around a circular path in a park that is $\frac{3}{4}$ miles long. If he walks $2\frac{1}{2}$ miles before he rests, how many times around the path did he travel?	Joel is walking around a circular path in a park. If he can walk $2\frac{1}{2}$ miles in $\frac{3}{4}$ of an hour, how far can he walk in an hour, assuming he walks at the same speed?
Guiding question for interpreting $2\frac{1}{2} \div \frac{3}{4}$	How many groups of $\frac{3}{4}$ are in $2\frac{1}{2}$?	If $\frac{3}{4}$ of a group gets $2\frac{1}{2}$, how much does a whole group get?
Meaning of reciprocal	The reciprocal $\frac{4}{3}$ means there are $\frac{4}{3}$ groups of $\frac{3}{4}$ in 1.	The reciprocal $\frac{4}{3}$ is the operator necessary to shrink $\frac{3}{4}$ to $\frac{1}{4}$ and then expand $\frac{1}{4}$ to 1.
Reason for multiplying the dividend by the reciprocal of the divisor	There are $\frac{4}{3}$ groups of $\frac{3}{4}$ in 1. There are $2\frac{1}{2}$ times as many groups of $\frac{3}{4}$ in $2\frac{1}{2}$ as there are in 1. Thus, there are $2\frac{1}{2}$ x $\frac{4}{3}$ groups of $\frac{3}{4}$ in $2\frac{1}{2}$.	Since we shrink/expand $\frac{3}{4}$ by $\frac{4}{3}$ to get 1 whole group, we have to shrink/expand $2\frac{1}{2}$ by $\frac{4}{3}$* in order to find out how much the whole group gets.

(Reproduced from Siebert, 2002, p. 254)

*Note: typo in the book ($\frac{3}{4}$ is written instead of $\frac{4}{3}$)

Li (2008) also summarizes the meaning of the division as "the same as the meaning of "division of whole numbers." It is an inverse operation of multiplication; that is, given the product of two numbers and one of these two numbers, find the other number" (p. 548). Put another way, he explains that, "Consistent with the approach to division of whole numbers, division of fractions is explained as a method for figuring out the number of times that a divisor can be measured out of the dividend" (p. 548). Two helpful diagrams show these connections between division with fractions and division with whole numbers.

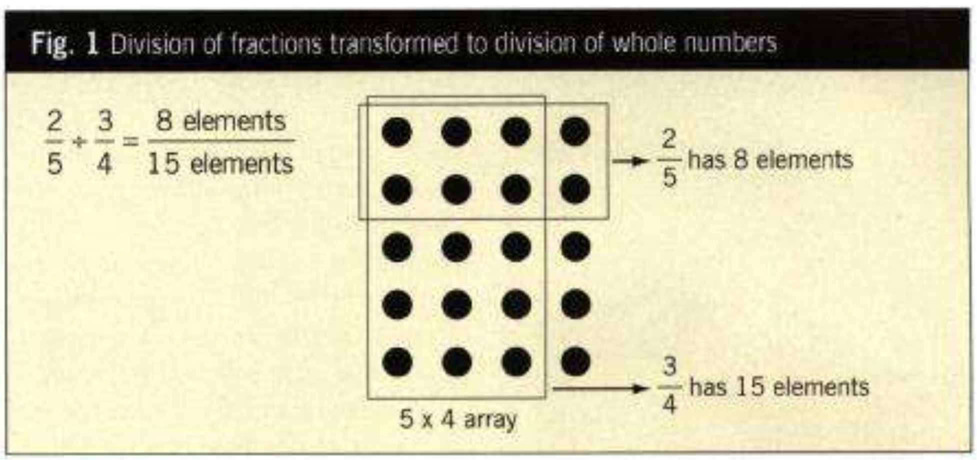
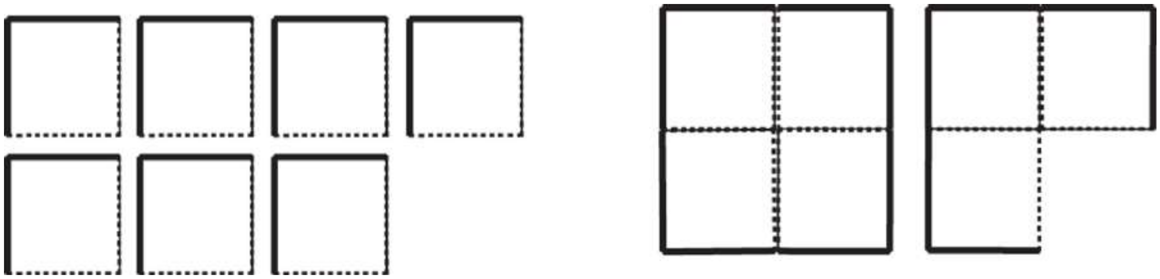


Figure 9 Division of fractions transformed to division of whole numbers

(Li, 2008, p. 548)

Task used to relate the meaning of fraction division to whole-number division



1. If everyone eats one-fourth of a square, seven persons will eat one and three-fourth squares.

   $\frac{1}{4} \times 7 = 1\frac{3}{4}$ (pizza)

2. If one and three fourth squares are equally shared among seven persons, everyone will get one-fourth of a square.

   $1\frac{3}{4} \div 7 = \frac{1}{4}$ (pizza)

3. If one and three fourth squares are given out as one-fourth of a square per person, seven people will get a share.

   $1\frac{3}{4} \div \frac{1}{4} = 7$ (persons)

Summary: The meaning of "division of fractions" is the same as the meaning of "division of whole numbers." It is an inverse operation of multiplication; that is, given the product of two numbers and one of these two numbers, find the other number.

Adapted from Li, 2008, p. 548.

Figure 10 Division of fractions transformed to division of whole numbers 2

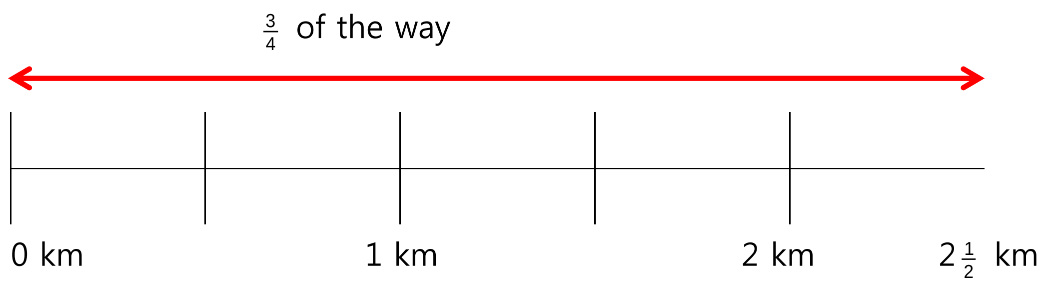
Effective teaching includes multiple and carefully selected representations for multiplying and dividing fractions

After working with a group of teachers, Peck and Wood (2008) recognized the importance of being able to teach – and respond to – a variety of representations in mathematics: "Students and teachers alike must be able to explain the mathematics and express the situation with symbols, charts, graphs, and diagrams, which are all ways of communicating mathematically" (p. 210). Equal groups, multiplicative comparison, repeated-subtraction/measurement, and fair-sharing/partitioning are all classifications for representing multiplication and division of fractions that are discussed by Peck and Wood (2008). In increasing their own flexibility with representations, teachers are better able to teach their students a variety of ways to solve problems in mathematics (Peck & Wood, 2008). Of equal importance is the careful selection of representations that are appropriate to the context of the problem.

In an effort to help teachers understand both *how* and *why* fraction division works, Cengiz and Rathouz (2011) suggest using stories, diagrams and symbols to: (i) see characteristics of each type of fraction operation; (iI) recognize differences between them; and, (iii) develop an understanding of when, and in which context, to use each of them. The authors reported on two teachers engaged in challenging fractions tasks where they had to move between representations (from "stories to diagrams and symbols" and from "symbols to stories and diagrams"), a helpful strategy in making sense of division with fractions. The authors suggest that activities be selected to provide experiences which help build the concept of the unit and connect representations.

Consider the following example, where a story is represented first as a diagram and then as symbols:

Story	George has driven $2\frac{1}{2}$ kilometers to get to his sister's house, but that he is only $\frac{3}{4}$ of the way. How many kilometers is the total distance to his sister's house?

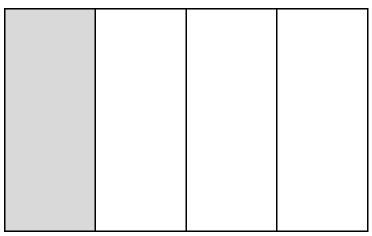
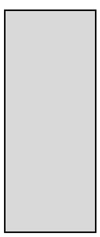
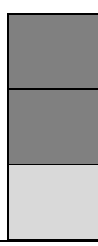
Diagram	
Solution in symbols	To find out how far the extra $\frac{1}{4}$ of the way is: • Divide $\frac{3}{4}$ of the way ($2\frac{1}{2}$ km) by 3 to get the distance for $\frac{1}{4}$. In symbols: $2\frac{1}{2} \div 3 = \frac{5}{6}$. Therefore $\frac{1}{4}$ of the way is $\frac{5}{6}$ • Multiply $\frac{1}{4}$ of the way by 4 to get $\frac{4}{4}$ (the whole way) $(2\frac{1}{2} \div 3) \times 4$* *Note: dividing by 3 and multiply by 4 is the same as solving $2\frac{1}{2} \times \frac{4}{3}$ (i.e., multiplying by the reciprocal of $\frac{3}{4}$)

(Reproduced from from Cengiz & Rathouz, 2011, p. 151)

Cengiz and Rathouz (2011) also draw our attention to the importance of tracking the unit when solving fraction problems and to recognize the relationship between, and roles of, the numerator and denominator. For example, in the case of $6 \div \frac{2}{5}$, it is helpful to think both in terms of the unit fraction ($\frac{1}{5}$), and to recognize that the numerator indicates "the number of groups shown" (2) and the denominator identifies "the number of equal groups in the whole batch" (5) (p. 148). Such "problems that require students to consider both division interpretations, attend to appropriate referent units, and form connections among representations promote a foundation for understanding fraction division" (p. 152).

Tools that prove particularly helpful for student learning include the number line as well as paper folding. In a study of multiplication with fractions conducted with Grade 6 students, Wyberg, Whitney, Cramer, Monson & Leavitt (2011) discuss the benefits of paper folding and number lines as helpful tools in multiplying fractions and highlight the importance of models that connect symbols and contexts. In this teaching experiment study, paper folding and the number line helped sixth grade students gain understanding beyond the fraction multiplication algorithm which, despite being a relatively simple procedure, is often not well conceptualized. “The paper model clearly showed students that the product of two fractions less than 1 is less than both fractions in the problem” (p. 292). In their study, “Many of the students explained that they knew they had the correct answer when the folded paper, the results of the algorithm, and the drawing of the number line all matched” (p. 294).

How to model $\frac{2}{3}$ *x* $\frac{1}{4}$ *using paper folding*

Step 1		Students fold the paper into four equal-size pieces and shade in one piece. The *whole* is the entire piece of paper
Step 2		Students then fold the paper so that only $\frac{1}{4}$ is showing. Since the remaining (unshaded) portion is hidden, it is easy to see the *unit* ($\frac{1}{4}$)
Step 3		Students then partition the $\frac{1}{4}$ into thirds and shade $\frac{2}{3}$ of the $\frac{1}{4}$ piece.

Step 4		Students can then unfold the paper to show the whole, and extend the horizontal thirds across the entire piece of paper (making equal sized smaller rectangles). The dark shaded squares then represent $\frac{2}{12}$ (the product of $\frac{2}{3}$ x $\frac{1}{4}$).

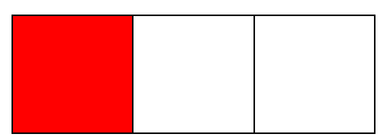
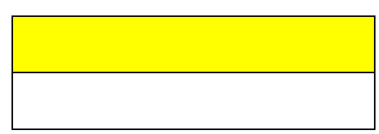
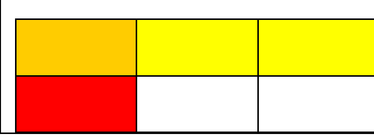
(Reproduced from Wyberg, Whitney, Cramer, Monson & Leavitt, 2011, p. 291)

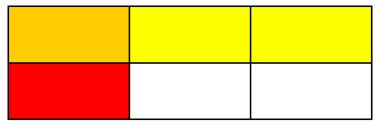
Noparit and Saengpun (2013) consider the number line as a tool to teach multiplication and division of fractions in Japan. Two teacher candidates and their grade six students participated in lesson study focusing on multiplying and dividing fractions. Lessons used by the teacher candidates were based on Japanese textbooks and the proportional number line was used as a tool to help students and teachers interpret and solve problems. The number line representation helped students think about fraction calculations in several different ways, and they made connections between them. Both students and their teachers had a more developed understanding of the calculations they were doing when they used the proportional number line as a tool.

The number line (along with unit fractions and paper folding) also emerged as key tools for learning operations with fractions in a detailed study conducted by Keijzer and Terwell (2001). Similarly, in a study by Siegler, Thompson and Scheider (2011) that examined connections between whole number and fraction development in twenty-four 11- and twenty-four 13-year-olds, the “mental number line” was a helpful tool for students to use when learning to understand fractions magnitudes. In number line tasks that included placing fractions and whole numbers on number lines to compare magnitude, the researchers also concluded that “Emphasizing that fractions are measurements of quantity might improve learning about fractions" (p. 293).

De Castro (2008) examined the role of cognitive models as the "missing link" to the learning of fraction multiplication and division with two sections of pre-high school students in the Bridge Program (where the students could receive additional help in math, science and English before entering high school). The students in this study initially expressed negative attitudes about learning fractions. However, using the models presented in the table below, students, who had prior knowledge of the cancel-and-multiply and invert-and-multiply procedures were able to make sense of what they were doing. "The use of cognitive models helped students understand the algorithm better and relate it to their schema, thus achieving greater retention." (p. 109) (As researchers ourselves, we can see how these models may be helpful. However, it is easy to anticipate a criticism that these too could become overly procedural, and caution that use of models should include opportunities for students to engage in creation of the models and meaning-making, through open problems, exploration and inquiry.)

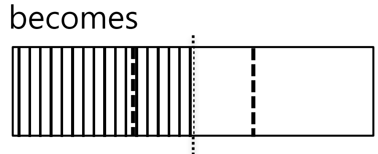
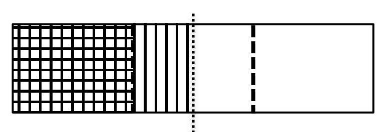
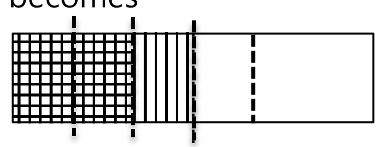
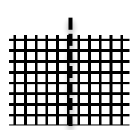
Fraction Multiplication Process: Using the cognitive model

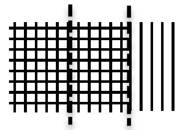
Sub-goals	Prompter	Representation/Output
1. Identify the multiplicand	$\frac{1}{3}$ x $\frac{1}{2}$	$\frac{1}{3}$ is the multiplicand
2. Draw a representation with vertical divisions	Shade the portion representing $\frac{1}{3}$ in a rectangular figure	
3. Identify the multiplier	$\frac{1}{3}$ x $\frac{1}{2}$	$\frac{1}{2}$ is the multiplier
4. Draw a representation with horizontal divisions	Shade the portion representing $\frac{1}{2}$ in a rectangular figure	
5. Superimpose the two rectangles		

6. Count double shaded/orange regions (numerator)		There is only 1 double shaded region
7. Count total number of regions (denominator)		There is a total of 6 regions in the figure
8. Represent the product	1 as the numerator and 6 as the denominator	The product of $\frac{1}{3}$ x $\frac{1}{2}$ is $\frac{1}{6}$

(Reproduced from de Castro, 2008, p. 105)

Fraction Division: Using the cognitive model

Sub-goals	Prompter	Representation/Output
1. Identify the dividend	$\frac{1}{3} \div \frac{1}{2}$	$\frac{1}{3}$ is the dividend
2. Draw a representation	Shade the portion representing $\frac{1}{3}$ in a rectangular figure	
3. Identify the divisor	$\frac{1}{3} \div \frac{1}{2}$	$\frac{1}{2}$ is the divisor
4. Identify the region of the divisor on the same figure	Shade the region representing $\frac{1}{2}$ in a rectangular figure	 becomes 
5. Superimpose and compare the double shaded with the single shaded regions	These regions must be of the same size	 becomes 
6. Count the number of double shaded regions (numerator)		There are 2 double shaded regions in the figure

7. Count the number of all shaded regions (denominator)		There is a total of 3 shaded regions in the figure
8. Represent the quotient	2 as numerator and 3 as denominator	The quotient of $\frac{1}{3} \div \frac{1}{2}$ is $\frac{2}{3}$

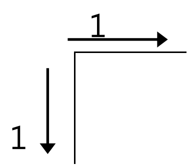
(Reproduced from de Castro, 2008, p. 106)

Specific suggestions for understanding multiplication with fractions

Multiplication is a complex operation. Depending on the size of the mutiplier, multiplication may result in increased, decreased or preserved quantities. Azim (2002) found that students perceive multiplication as either repeated addition of whole numbers or division (fractions), depending on whether the quantity increases or decreases in size, respectively. Azim identified a number of methods that help develop an understanding of multiplication, in all its forms. Making real-life connections, for example, to photocopying (where a photocopier can preserve, reduce or enlarge an image, depending on the multiplier) or increasing/decreasing fractional recipe amounts can help students conceptualize how the multiplication of fractions will result in a smaller quantity.

As previously discussed, we know that based on early experience of operations with whole numbers, students tend to persist in a strongly held belief and overgeneralization that addition and multiplication operations always produce a larger total or quantity, and when you subtract or divide, you get a smaller quantity. Graeber and Campbell (1993) reported on student misconceptions and provided suggestions for helping students figure out how the opposite can be true, in the case of multiplying and dividing fractions.

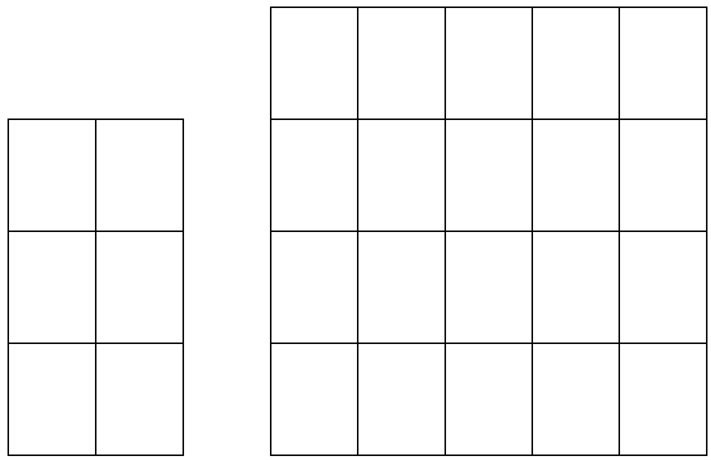
One of the common interpretations of multiplication – repeated addition – does not lend itself well to the multiplication of rational numbers (e.g., mixed numbers, proper fractions or decimals). Therefore, students should have more of a sense of multiplication than just repeated addition. The area model of multiplication is helpful for seeing how “multiplication makes smaller.” When two fractions of an area are laid on top of each other, the overlapping section is the answer. This is very similar to the explanation provided in Section 2 of this literature review, which discusses the concept of ‘shared space’ and has proven to be a particularly powerful representation for helping teachers and students make sense of fraction multiplication.



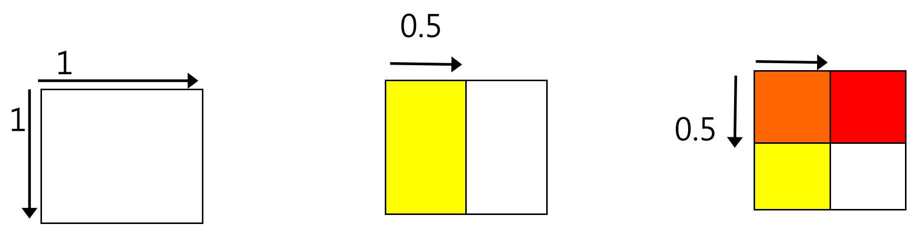
In an area model, one factor describes the width of a rectangle, the other factor describes the length of the rectangle.

The product is modeled by the area.

a) Area model for 3 x 2 and 4 x 5



b) Area model for 0.5 x 0.5 is 0.25



(Reproduced from Graeber & Campbell, 1993, p. 409)

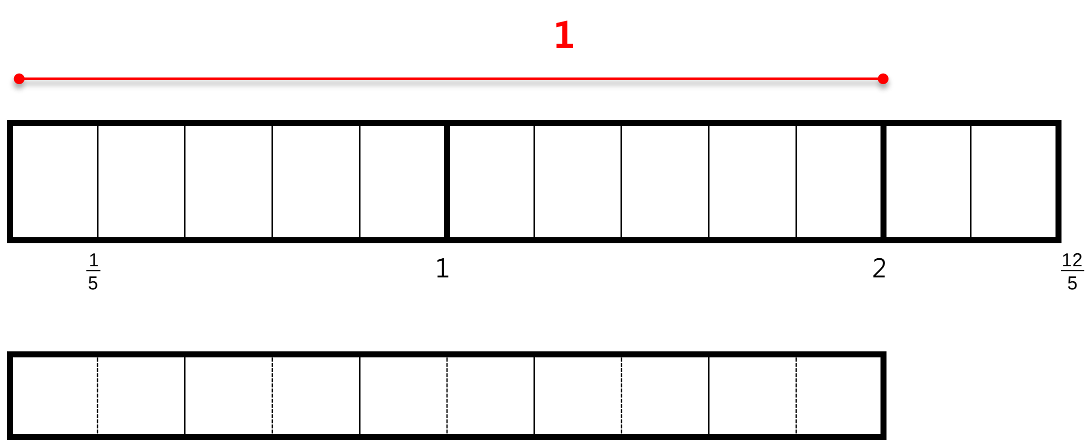
Another strategy in understanding how fraction multiplication results in a smaller quantity is to identify the pattern in a list of multiplication facts beginning with whole numbers and then continuing to fractions less than 1 (i.e., 3x10, 2x10, 1x10, 0x10, $\frac{1}{2}$x10, etc.) and recognize that not all of these results in a bigger number (Graeber & Campbell, 1993).

Specific suggestions for understanding division with fractions

The measurement model can help students make sense of "division making bigger," even before considering formal algorithms for fraction division. Graeber and Campbell (1993) suggest that "fraction pieces, drawings, or their [students'] knowledge of the monetary system" are helpful in solving division problems and using rational numbers that result in the answer being made bigger (p. 410).

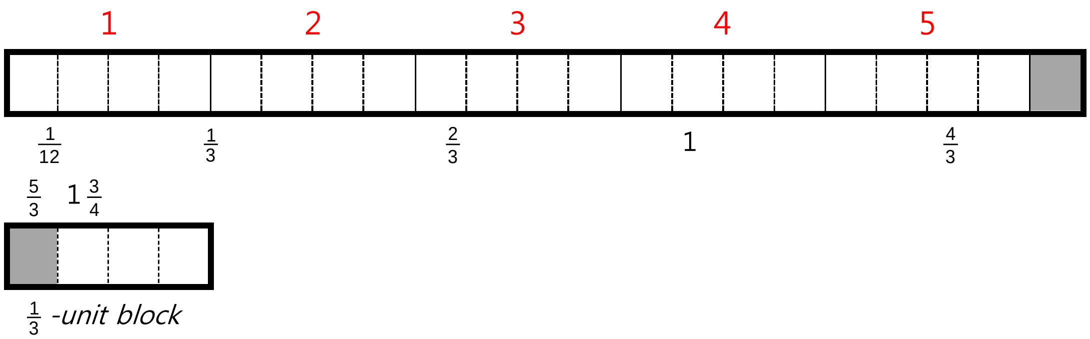
The measurement model and the repeated-subtraction model prove useful in eliminating fraction division misconceptions. "The measurement model can be used to reveal the relationship between the answer and the divisor. This model can also build connections with other concepts and models, such as the division algorithm, remainders, and the missing-factor approach" (Coughlin, 2010, p. 283). Using repeated subtraction, as shown in the below figure, better highlights the relationship between the answer and the divisor than simply solving with the invert-and-multiply rule.

When calculating $\frac{12}{5} \div 2$, the $\frac{2}{5}$ left over from the subtraction is $\frac{1}{5}$ of the 2-unit (or $\frac{10}{5}$-unit) block.



2-unit (or $\frac{10}{5}$ -unit) block

For $1\frac{3}{4} \div \frac{1}{3}$, repeated subtraction demonstrates that the remainder is $\frac{1}{4}$ of a $\frac{1}{3}$-unit block.



(Reproduced from Coughlin, 2010, p. 285)

Unpacking a conceptual lesson on dividing fractions, Philipp (2000) also advocates for a measurement approach, arguing that it is "difficult to conceptualize $1 \div \frac{4}{5}$ using a partitive model. How might we share a cup of sugar among $\frac{4}{5}$ people? For fraction situations, it is often more meaningful to use a measurement model: If we had 1 cup of sugar and each recipe called for $\frac{4}{5}$ cup of sugar, then how many recipes could we bake?" (p. 11).

7. Recommendations

Although the body of research on multiplication and division with fractions is substantially smaller than that for general fractions research, there are some clear directions for future activity based on this literature review. The recommendations are classified into two broad categories: recommendations for supporting high quality teaching of fractions operations, and recommendations for supporting high quality research on fractions operations.

Supporting High Quality Teaching of Fractions Operations

- Given the particularly troublesome research findings on pre-service teacher understanding of fractions operations, combined with the reported lack of attention to this content in education programs, it would be beneficial to consider ways to increase the fractions content learning in pre-service programs. This could involve combined efforts of faculty who are mathematics educators and Ministry staff to consider innovative ways to support teacher candidates in their learning, including opportunities to benefit from online resources of the EduGAINS site such as the Fractions Digital Paper, CLIPS fractions, and other related online fractions resources.
- Similarly, current in-service teachers require professional learning opportunities and high quality resources that are specifically focused on content learning of, and related effective pedagogies for, developing deep student understanding of multiplication and division with fractions. This includes an emphasis on the conceptual underpinnings that support effective fractions teaching and sustained student learning. District school board educators along with Ministry of Education personnel could collaborate in these efforts using a multipronged and multi-modal strategy (some online, some face-to-face, a variety of offerings of short and longer duration, continued development of, and implementation of, high quality tasks and lesson materials for teachers).
- As a way forward, educators can review the current best practices described in Section 6 of this literature review.

Supporting High Quality Research on Fractions Operations

Just as there are recommendations for improving the teaching of fractions, there are also recommendations for improving the research on multiplication and division with fractions. These include:

- Supporting and implementing additional and more robust research at the in-service level where the literature is relatively sparse.
- Continuing to develop the Fractions Learning Pathways Framework to include multiplication and division with fractions, and continue developing field-testing and disseminating accessible high caliber resources for teachers related to the Fractions Learning Pathways Framework.
- Engaging in more research on effective teaching of multiplication and division with fractions, including seeking answers to questions such as the following:
    - How can we tap into students' informal knowledge and early strategies with fair sharing to help them build deep conceptual understanding that later increases understanding of more formalized fractions operations?
    - How can the unit fraction help with the teaching and learning of multiplication and division of fractions? How can unit fractions help students to understand equivalent fractions and common denominators (which are central to multiplication and division with fractions)?
    - Which types of tasks, contexts and representations support Ontario students most effectively in developing their understanding of multiplication and division with fractions?

To pursue any of the above recommendations for research, once again, partnerships will be central to success. Researchers, teachers, and mathematics leaders would benefit from working together to gain clear and well-informed answers to some of these questions.

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