TIPS4RM

Leading MATH Success

TIPS4RM Targeted Implementation and Planning Supports for Revised Mathematics

Mathematical Processes

Mathematical Processes

Problem Solving

Reasoning and Proving

Reflecting

Selecting Tools and Computational Strategies

Connecting

Representing

Communicating

Context – Why it is important to engage students in mathematical processes

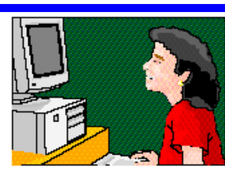
Even if you've stopped growing physically, you certainly haven't stopped growing mentally and emotionally. Nor do you stop learning after you finish school, not as long as there are opportunities for learning and growth all around us. Learning also comes in many—and often surprising—forms. But no matter how it appears, learning is forever and learning is for the future.

(*Ontario Prospects 2002: Ontario's Guide to Career Planning*)

It is important that students see mathematics as sensible, useful, and doable. Teachers should take every opportunity during the instructional/learning process to help students develop a positive disposition towards mathematics. By focusing on mathematical process skills, teachers empower students mathematically.

Context Connections

Primary/Junior



Intermediate/Senior

Next Steps

Mathematical processes develop through different grade levels and support lifelong learning.

Visual

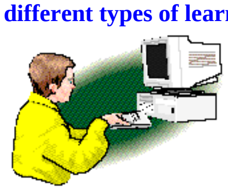
Auditory

Kinaesthetic

Mathematical processes are taught and assessed in ways that address the different needs of different types of learners.

Guided

Shared



Independent

A variety of groupings and instructional strategies help students improve their mathematical processes.

Mathematical Process Expectations

The seven mathematical process expectations describe the actions of doing mathematics. They support the acquisition and the use of mathematical knowledge and skills. They can be mapped to three of the categories of the Achievement Chart – Thinking, Communication, and Application. The fourth category, Knowledge and Understanding, connects to the content of each course/program. Students apply the mathematical processes as they learn the content for each course/program.

Problem Solving

The Ontario Curriculum, Mathematics, 2005

Students will develop, select, apply, and compare a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding.

Problem solving is central to learning mathematics. It forms the basis of effective mathematics programs and should be the mainstay of mathematical instruction. Problem solving is considered an essential process through which students are able to achieve the expectations in mathematics, and it is an integral part of the mathematics curriculum in Ontario.

Role of Students	Instructional Strategies
Planning • Understand the problem • Try different techniques and strategies • Generate some examples • Ask thoughtful questions **Collecting data related to the problem** • Take and record measurements • Search the Internet for secondary data • Check that data being gathered is appropriate to the inquiry at hand **Selecting and applying a problem-solving strategy** • Include some of the following strategies: – draw a diagram or picture – make a simpler but similar problem – act it out – create a mathematical model – work backwards – use a formula – look for a pattern – guess and check – make and state assumptions – make a scale drawing – make an organized list – use logical reasoning – consider alternative strategies and/or blend strategies – monitor progress and revise, as necessary – ask if the answer is reasonable – consider extensions and variations to the problem and the solution • Incorporate different strategies over time	• Collaborate with students, asking questions or thinking aloud when a student or a group of students is not making progress. • Scaffold based on knowledge and skills of individual students. • Provide resources and time for students to gather data, detect patterns, make and justify conjectures. • Ask probing questions if data or strategy seems to be unconnected or inappropriate to the inquiry. • Organize pooling of data, as appropriate. • Guide students as they apply their chosen strategy. • Facilitate the purposeful sharing of different problem-solving strategies for the same problem. • Direct students to use multiple strategies to solve the same problem, when appropriate. • Recognize, encourage, and applaud perseverance. • Encourage students to work on tasks that demand sustained effort over time, e.g., problem of the week. • Validate different approaches to the same problem. • Discuss the relative merits of different strategies for specific types of problems. • Use cross-curricular applications to demonstrate the usefulness of mathematics. • Support and encourage risk taking, and applaud creative approaches. • Encourage independence and interdependence. • Facilitate the sharing of student findings. • Model alternative procedures and strategies, such as using manipulatives and technology.
Sample Questions	**Sample Feedback**
• How does this problem remind you of a problem you have solved before? • What are the connections between this problem and [identify the problem] we solved last week? • What are some specific cases in this problem? • How would you state this problem in your own words? • What problem-solving strategies have you tried? • What strategy will you try next? • What were the advantages and disadvantages of the strategies you tried? • Which strategies can you combine to help you solve this problem? • What factors make this a difficult problem? • What are some of the complexities of this problem?	• Consider exploring [identify the student]'s idea. • Take a few minutes to talk with other groups. I'll be back to see how you're progressing. • Think about how you can apply this strategy more efficiently. • Consider some specific cases first. • Find someone who has used a different strategy to solve this problem and talk about your approaches. In a few minutes, we'll discuss what you learned. • Please explain the strategy you used. • How does this relate to the problem? • Reread the problem to identify the most important aspects and facts to consider. • Reread the problem and consider a different perspective.

Reasoning and Proving

The Ontario Curriculum, Mathematics, 2005

Students will develop and apply reasoning skills (e.g., recognition of relationships, generalization through inductive reasoning, use of counter-examples) to make mathematical conjectures, assess conjectures and justify conclusions, and plan and construct organized mathematical arguments.

Students make sense of mathematics through reasoning. An organized, analytical, well-reasoned approach to learning mathematical concepts and processes and to solving problems requires an emphasis on reasoning.

Role of Students	Instructional Strategies
Hypothesizing/making conjectures • Combine given information with intuition to make a reasoned guess when prompted • Refine hypothesis as evidence is gathered • Make a reasoned guess as to: – the answer – the strategy likely to lead to a solution – where in the process and/or why an attempted solution failed **Making inferences, conclusions, and justifications** • Use models and logic to infer/conclude • Adjust models, as needed • Reason inductively by considering specific cases and identifying patterns • Analyse and evaluate the mathematical thinking and strategies of others, orally or in writing • Present arguments in a logical and organized manner • Include enough detail and clarity that the reader/ listener can follow their thinking • Try multiple examples, e.g., make multiple trials using a GSP® sketch; make systematic trials using manipulatives or pencil and paper • Look for a case that does not work, i.e., a counter-example • Recognize the characteristics of an acceptable argument/proof • Follow and understand an argument presented by someone else	• Ask questions that require students to hypothesize and make conjectures, e.g., What if…? • Facilitate sharing of hypotheses/conjectures and the reasoning behind them. • Accept all student suggestions and help them decide what evidence they need to confirm or refute their hypotheses. • Model how to adjust a hypothesis that has been refuted by evidence. • Nurture risk taking, e.g., thinking out loud, making a hypothesis that may be false. • During whole-class discussions, foster behaviours such as active listening to the reasoning of others; legitimizing errors as part of the learning process; and tolerating ambiguity. • Provide frequent opportunities for students to work in small, mixed-ability groups so that students who are experiencing difficulty can hear and see the reasoning and proofs of their peers. • Provide frequent opportunities for students to work in homogeneous groups so that differentiated instructional activities target their readiness to reason in different ways, e.g., algebraic, inductive, deductive. • Listen to what students say and look at what students write to identify misunderstandings and misconceptions, and then differentiate instruction accordingly. • Recognize, model, and develop a mathematical style of dialogue and argument in the classroom. • Provide opportunities for students to read, hear, question, and discuss explanations of others. • Lead students to make generalizations after repeated trials, and by identifying patterns. • Provide and ask students to give counter-examples, explaining what this means in terms of the conjecture. • Provide students with one or more numerical examples and parallel these with the generalization e.g., the development of a formula. • Ask students to explain the reasoning that accompanies each step of a mathematical argument or proof.

Reasoning and Proving

Sample Questions	Sample Feedback
• How can we show that this is true for all cases? • In what cases might our conclusion not hold true? • How can we verify this answer? • Explain the reasoning behind your prediction. • Why does this work? e.g., the procedure for bisecting an angle using compasses • Explain the reasons for your answer. • Consider the pattern. What do you think will happen if this pattern continues? • Show how you know that this statement is true. • Give an example of when this statement is false. • Explain why you do not accept the argument as proof. • How could we check that solution? • What other situations need to be considered?	• What you have presented is true for the cases you considered. Explain some cases where this situation will not be true. • Present your solution, showing all the steps so someone else will understand your thinking. • Your reasoning was good to this point in your argument. Study your next point to see if you can identify the flaw in reasoning. • Identify the flaw in this argument. How would you correct it? • If we accepted this reasoning as true, it would mean that…. However, we know that…is not true. Therefore, there must be a flaw in the original reasoning. Let's search out the flaw together by going over the argument step by step. • Describe your thinking in more detail so your argument can be followed. • How does this reasoning follow from what you said? • Before you implied…and now you are saying…. How can both be true?

Reflecting

The Ontario Curriculum, Mathematics, 2005

Students will demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions).

Students become good problem solvers when they regularly and consciously reflect on and monitor their own thought processes.

Role of Students	Instructional Strategies
Considering data collected • Search for relevant primary and secondary data • Check that data being gathered is appropriate to the inquiry **Reflecting on new skills, concepts, and questions to see how they connect to prior knowledge** • Apply and extend knowledge to new situations • Examine questions and demonstrate flexibility in choice of strategy based on the nature of the question • Verify a solution to a problem by using a different method • Consider the reasonableness of their answer • Self-monitor progress while problem solving and revise, as necessary • Propose alternative approaches to a problem	• Ask students to think about their thinking, e.g., How do you know? How is this similar to a problem you have done before? • Model how to use the reflective process, explicitly stating what you are doing, e.g., As I reflect on your explanation, I…; As I think about this problem, I am reminded of…. • Generate cognitive dissonance in students who have misunderstood a concept so they can change their understanding. • Use graphic organizers such as Venn diagrams and concept maps to review and summarize concepts. • Encourage students to ask themselves "what-if" questions.
Sample Questions	**Sample Feedback**
• Have you thought about…? • What do you notice about…? • What patterns do you see? • Does this problem/answer make sense to you? • How does this compare to…? • What could you start with to help you explore the possibilities? • How can you verify this answer? • What evidence of your thinking can you share? • Is this a reasonable answer, given that…?	• Explain how the data you collected to inform your thinking connects to the problem. • I can follow your thinking up to here. How can you help me understand your next ideas? • Share your explanation with this group and consider their feedback as you revise your work. • How does this all make sense together? • What solution could be more suitable? • How is this result applicable to the problem?

Selecting Tools and Computational Strategies

The Ontario Curriculum, Mathematics, 2005

Students will select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems.

Students need to develop the ability to select the appropriate electronic tools, manipulatives, and computational strategies to perform particular mathematical tasks, to investigate mathematical ideas, and to solve problems.

Selecting Tools

Role of Students	Instructional Strategies
• Understand when mental arithmetic or a pencil-and-paper calculation or estimation is more appropriate than technology • Use an appropriate tool when: – an exact answer is needed – computation involves several numbers or numbers with more than one digit – the numbers are not easily calculated mentally • Use technology (e.g., graphing calculators, spreadsheets, The Geometer's Sketchpad®, Fathom®, Tinkerplots®) to explore, gather, display, manipulate, and present data in a variety of ways • Use manipulatives and/or technology to develop understanding of new concepts, for communicating, or for performing certain tasks	• Introduce new technology/manipulatives in ways that allow students to explore and build confidence, e.g., students work in pairs. • Model the use of tools, e.g., using a large model, overhead, or a data projector. • Demonstrate situations in which the new technology/manipulative would be an appropriate choice of tool. • Encourage students to use technology to solve problems when the focus is on problem solving rather than on pencil-and-paper skills. • Make available a range of tools for students to use during instruction and assessment. • Present a range of tasks that promote the use of technology. • Have students use technologies to investigate multiple examples quickly before generalizing relationships, e.g., GSP® and spreadsheets. • Ask students to use tools to demonstrate and communicate their understanding of a skill or concept. • Demonstrate how selected tools can be used to help form conjectures, make decisions, and/or solve problems.
Sample Questions	**Sample Feedback**
• How did the learning tool you chose contribute to your understanding/solving of the problem? assist in your communication? • In what ways would [name a tool, e.g., GSP®, a graphing calculator, linking cubes, a ruler] assist in your investigation/solving of this problem? • What other tools did you consider using? Explain why you chose not to use them.	• You have chosen a different tool than other students. Why did you choose this one? • You have selected an appropriate tool but an error has occurred. Review your procedure to identify the error. e.g., key entry on a calculator, improper use of features such as constructing versus drawing in GSP® • Share your solution with someone who has used a different tool, and discuss the merits of each. • Use [name a tool] and see how it helps you solve the problem.

Selecting Computational Strategies

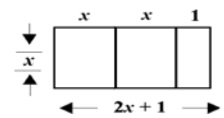
Role of Students	Instructional Strategies
• Perform mental calculations, e.g., estimate by substituting rounded values into formulas; add many large numbers, keeping track of rounding error to adjust the total; multiply by rounding and considering place value; apply the distributive property • Develop and use a personal set of referents for measurement, e.g., 1 cm is approximately the width of a baby finger • Estimate using properties of numbers, e.g., a square root from known square numbers • Select different computational strategies depending on the numbers involved, e.g., 25×16, 23×16, 19×16	• Provide regular opportunities for students to further develop and apply mental mathematics and estimation skills. • Model different mental strategies. • Lead students to understand standard referents, by having them share their own referents when explaining their reasoning. • Model different computational strategies, and explain why you choose to use them.
Sample Questions	**Sample Feedback**
• Explain why you chose this computational strategy. • Is an exact answer necessary for this question? Would estimation be adequate? Explain. • Think of a different way to do the calculation that may be more efficient. • What estimation strategy did you use? Was your result sufficiently accurate for the question? • Why was a calculator necessary (or helpful) for this problem? • Explain another computational strategy that could be used for this problem.	• Please model the computational strategy you used for a classmate. • Your choice of strategy has shown me a connection I had not thought of before. Now I see that…. • Please think aloud while you apply the strategy so I can learn why you are getting this answer. • Try another computational strategy and see if your result is the same.

Connecting

The Ontario Curriculum, Mathematics, 2005

Students will make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports).

Students need to see the connections and the relationships between mathematical concepts and skills from one strand of mathematics to another. As they continue to make such connections, students begin to see that mathematics is more than a series of isolated skills and concepts and they can use their learning in one area of mathematics to understand another. Seeing connections among procedures and concepts also helps deepen students' mathematical understanding. Further, making connections between the mathematics they study and its application in their everyday lives helps students see how useful and relevant it is in the world beyond the classroom.

<table>
<tr><th>Role of Students</th><th>Instructional Strategies</th></tr>
<tr><td>
• Apply a strategy or reference system that draws on previous learning in another context
• Make connections between new and prior knowledge to make sense of what they are learning
• Apply mathematics to contexts outside of mathematics
• Use different models to best convey mathematical information and demonstrate their conceptual understanding of a procedure
• Make connections between different representations, e.g., numeric, graphical, and/or algebraic
</td><td>
• Activate prior knowledge when introducing a new concept in order to make a smooth connection between previous learning and new concepts.
• Introduce skills in context to make connections between particular manipulations and problems that require them, e.g., "factor" connects to finding the dimensions of a rectangle whose area has been given.


• Make explicit links between mathematical concepts and skills and those in other disciplines.
• Allow students to explore their own procedures and algorithms, monitoring these for correctness.
• Integrate strands, explicitly demonstrating and reinforcing connections.
• Use visuals to connect procedures and concepts, e.g., graph of a line to find the *x*-intercept.
• Make connections within mathematics explicit, e.g., notice the similarity between finding the *x*- and *y*-intercept in a linear relation, and finding the *x*- and *y*-intercept in a quadratic relation.
• Make explicit connections between mathematics and everyday student experiences, e.g., My neighbour used his understanding of geometry to finish making his skateboard ramp in this way….
• Be open and receptive to connections identified by students and recognize when a student makes a significant mathematical connection.
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<tr><th>Sample Questions</th><th>Sample Feedback</th></tr>
<tr><td>
• What other math have you studied that has some of the same principles, properties, or procedures as this? e.g., How does knowing the formula for the volume of the rectangular prism help us to find the formula for the volume of a triangular prism? How is adding "like terms" similar to adding integers?
• How do these different representations connect to one another? e.g., What seems to be the connection between the horizontal intercept of the graph and the numeric table of values?
• When could this mathematical concept or procedure be used in daily life?
• What connection do you see between a problem you did previously and today's problem?
</td><td>
• How can you relate your understanding of…to this problem?
• How does your representation (e.g., diagram, sketch, manipulative) connect to…, e.g., the algebraic solution?
• Please describe the connections you see between …and….
• How does this method relate to this problem?
• How is that thinking connected to the question?
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Representing

The Ontario Curriculum, Mathematics, 2005

Students will create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial; onscreen dynamic representations), connect and compare representations, and select and apply the appropriate representations to solve problems.

Students represent mathematical ideas and relationships and model situations using concrete materials, pictures, diagrams, graphs, tables, numbers, words, and symbols. Learning various forms of representation helps students to make connections and develop flexibility in their thinking about mathematics.

Role of Students	Instructional Strategies
• Select an appropriate representation and defend their choice: – physical/concrete/manipulative – electronically generated, e.g., graphs, dynamic geometry representation – mental image – numerical, e.g., table of values – graphical – scale drawing – diagram – graphical organizers, e.g., Venn diagram, T-chart, concept map – equation/algebraic expression/formula – algorithm/logic model • Understand that various representations can be used to appropriately represent the same situation • Understand that there may be different variations of one representation, e.g., algebraic expressions may be equivalent yet appear different • Understand the role of constants (e.g., pi) and variables (e.g., radius) in formulas and patterning rules (e.g., “double the previous term”) • Use multiple representations, as required	• Model various ways to demonstrate understanding, e.g., talking, writing, graphing, explaining, questioning, drawing, kinaesthetic movement. • Use grade-appropriate mathematical models when presenting or reviewing a concept. • Introduce new concepts using concrete materials. • Pose questions that require students to use different representations as they are working at each level of conceptual development – concrete → visual → symbolic – continually revisiting each stage to make connections. • Allow individual students the time they need to solidify their understanding at each conceptual stage.
Sample Questions	**Sample Feedback**
• What would other representations of this problem demonstrate? • Explain why you chose this representation. • How could you represent this idea algebraically? graphically? • Does this graphical representation of the data bias the viewer? Explain. • What properties would you have to use to construct a dynamic representation of this situation? • In what way would a scale model help you solve this problem?	• Show how you can represent this situation more efficiently. • How can your representation of the data include your outliers? • In what other way(s) can you represent this problem?

Communicating

The Ontario Curriculum, Mathematics, 2005

Students will communicate mathematical thinking orally, visually, and in writing, using mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

Students…[use] a variety of appropriate representations including numbers, symbols, pictures, graphs, diagrams, and words. Through communication, students are able to reflect upon and to clarify ideas, relationships, and mathematical arguments.

Role of Students	Instructional Strategies
• Use developmentally appropriate sentences and suitable mathematical vocabulary in responses, as appropriate for the audience, e.g., create separate sentences for separate ideas; avoid run-on sentences • Respond to instructions orally, in writing, and visually, as appropriate, e.g., explain, discuss, describe, justify, compare, suggest, write, tell, read, share, demonstrate, present • Use correct mathematical language and vocabulary in explanations, e.g., interpolate, extrapolate, draw a line of best fit, evaluate, factor, expand, simplify, solve, rearrange, drag a vertex, transform by reflection • Present thinking and arguments in a logical and organized manner • Respond clearly with sufficient detail so that thinking can be understood • Interpret and summarize information from charts and graphs, providing appropriate detail, e.g., describe patterns and contrasts • Use the symbolic language of mathematics correctly, e.g., use = down the left side when simplifying expressions and in between two equal expressions when solving equations; use $\angle$ for naming angles • Read and reread all of the given information and instructions to ensure understanding, e.g., identify key information needed to solve the problem • Communicate mathematical learning by combining various representations, e.g., words with diagrams, charts or graphs with verbal descriptions	• Use literacy strategies to help students make sense of what they read and see, e.g., anticipation guide, word wall, mind mapping. • Encourage students to use correct mathematical language and conventions, e.g., present arguments during group or class discussions, explaining solutions. • Introduce new terminology in a variety of ways, e.g., demonstrations, examples, definitions. • Coach students in proper usage of terminology and conventions, as needed. • Model the correct use of mathematical symbols, conventions, vocabulary, and notations. • Provide informal feedback on an individual student basis during the learning process, e.g., observations of correct or incorrect use of conventions. • Display samples of student work that exhibit the desired integration of narrative, symbolic, and graphic forms.
Sample Questions	**Sample Feedback**
• How can you express (explain, describe) this in a different way? • What is a definition for…? • How can you reword this question (answer)? • What mathematical symbols could you use to communicate this statement? • What mathematical operations are implied by the wording of this problem?	• How would you show the steps to your solution so someone else can follow your thinking? • If I follow your equal signs, it looks like 8 = 1. Since we know this is not true, how can we express this differently? • Use *Smart Ideas* to help you connect your ideas and communicate them more clearly. • Provide a more complete, organized outline of your ideas so that your reasoning can be followed. • How might you present this argument and data to the principal? • Be more specific and give more details. • Express this in a different way.

Connecting Mathematical Processes with the Achievement Chart

One way to visualize the interconnections among the categories of the Achievement Chart and the Mathematical Processes is shown in the gears below. This diagram illustrates the viewpoint that "Skills without conceptual understanding are meaningless; conceptual understanding without skills is inefficient. Without problem-solving skills, skills and conceptual understanding have no utility."

Mathematics Program Advisory, June 1996.

Knowing facts and procedures is an important aspect of mathematics education. At each grade level students learn basic facts – mental mathematics skills and the use of standard algorithms and procedures.

Conceptual understanding is another key component of mathematics education. Students with conceptual understanding see mathematics as a related whole. Applying and representing mathematical ideas in different ways for different situations, and connecting procedures and concepts are some indicators of conceptual understanding.

The mathematical processes are integral to problem solving. Students deepen their knowledge and understanding as they develop, refine, and use these processes in doing mathematics.

For further explanation and information, see the Continuum and Connections and the Assessment packages in the LMS electronic library.

Learners of all ages recursively construct meaning or knowledge based on their experiences and prior knowledge and understanding, and by applying mathematical processes. Students become resilient, flexible, competent, productive, and confident lifelong problem solvers.