

Ministry of Education



The Ontario Curriculum  
Grades 11 and 12

REVISED

# Mathematics

2007





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Une publication équivalente est disponible en français sous le titre suivant :  
*Le curriculum de l'Ontario, 11<sup>e</sup> et 12<sup>e</sup> année – Mathématiques, 2007.*

This publication is available on the Ministry of Education's  
website, at [www.edu.gov.on.ca](http://www.edu.gov.on.ca).

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# INTRODUCTION

This document replaces *The Ontario Curriculum, Grade 11: Mathematics, 2006*, and the Grade 12 courses in *The Ontario Curriculum, Grades 11 and 12: Mathematics, 2000*. Beginning in September 2007, all Grade 11 and Grade 12 mathematics courses will be based on the expectations outlined in this document.

## SECONDARY SCHOOLS FOR THE TWENTY-FIRST CENTURY

The goal of Ontario secondary schools is to support high-quality learning while giving individual students the opportunity to choose programs that suit their skills and interests. The updated Ontario curriculum, in combination with a broader range of learning options outside traditional classroom instruction, will enable students to better customize their high school education and improve their prospects for success in school and in life.

## THE IMPORTANCE OF MATHEMATICS IN THE CURRICULUM

This document provides a framework outlining what students are expected to know and be able to do by the end of each of the courses in the Grade 11–12 mathematics curriculum. The required knowledge and skills include not only important mathematical facts and procedures but also the mathematical concepts students need to understand and the mathematical processes they must learn to apply.

The principles underlying this curriculum are shared by educators dedicated to the success of all students in learning mathematics. Those principles can be stated as follows:<sup>1</sup>

- Curriculum expectations must be coherent, focused, and well-articulated across the grades.
- Learning mathematics involves the meaningful acquisition of concepts, skills, and processes and the active involvement of students in building new knowledge from prior knowledge and experience.
- Learning tools such as manipulatives and technologies are important supports for teaching and learning mathematics.
- Effective teaching of mathematics requires that the teacher understand the mathematical concepts, procedures, and processes that students need to learn, and use a variety of instructional strategies to support meaningful learning.
- Assessment and evaluation must support learning, recognizing that students learn and demonstrate learning in various ways.

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1. Adapted from *Principles and Standards for School Mathematics*, developed by the National Council of Teachers of Mathematics (Reston, VA: NCTM, 2000).

- Equity of opportunity for student success in mathematics involves meeting the diverse learning needs of students and promoting excellence for all students. Equity is achieved when curriculum expectations are grade- and destination-appropriate, when teaching and learning strategies meet a broad range of student needs, and when a variety of pathways through the mathematics curriculum are made available to students.

The Ontario mathematics curriculum must serve a number of purposes. It must engage all students in mathematics and equip them to thrive in a society where mathematics is increasingly relevant in the workplace. It must engage and motivate as broad a group of students as possible, because early abandonment of the study of mathematics cuts students off from many career paths and postsecondary options.

The unprecedented changes that are taking place in today's world will profoundly affect the future of today's students. To meet the demands of the world in which they live, students will need to adapt to changing conditions and to learn independently. They will require the ability to use technology effectively and the skills for processing large amounts of quantitative information. Today's mathematics curriculum must prepare students for their future roles in society. It must equip them with an understanding of important mathematical ideas; essential mathematical knowledge and skills; skills of reasoning, problem solving, and communication; and, most importantly, the ability and the incentive to continue learning on their own. This curriculum provides a framework for accomplishing these goals.

The development of mathematical knowledge is a gradual process. A coherent and continuous program is necessary to help students see the "big pictures", or underlying principles, of mathematics. The fundamentals of important skills, concepts, processes, and attitudes are initiated in the primary grades and fostered throughout elementary school. The links between Grade 8 and Grade 9 and the transition from elementary school mathematics to secondary school mathematics are very important in developing the student's confidence and competence.

The secondary courses are based on principles that are consistent with those that underpin the elementary program, facilitating the transition from elementary school. These courses reflect the belief that students learn mathematics effectively when they are given opportunities to investigate new ideas and concepts, make connections between new learning and prior knowledge, and develop an understanding of the abstract mathematics involved. Skill acquisition is an important part of the learning; skills are embedded in the contexts offered by various topics in the mathematics program and should be introduced as they are needed. The mathematics courses in this curriculum recognize the importance of not only focusing on content, but also of developing the thinking processes that underlie mathematics. By studying mathematics, students learn how to reason logically, think critically, and solve problems – key skills for success in today's workplaces.

Mathematical knowledge becomes meaningful and powerful in application. This curriculum embeds the learning of mathematics in the solving of problems based on real-life situations. Other disciplines are a ready source of effective contexts for the study of mathematics. Rich problem-solving situations can be drawn from related disciplines, such as computer science, business, recreation, tourism, biology, physics, and technology, as well as from subjects historically thought of as distant from mathematics, such as geography

and art. It is important that these links between disciplines be carefully explored, analysed, and discussed to emphasize for students the pervasiveness of mathematical concepts and mathematical thinking in all subject areas.

The choice of specific concepts and skills to be taught must take into consideration new applications and new ways of doing mathematics. The development of sophisticated yet easy-to-use calculators and computers is changing the role of procedure and technique in mathematics. Operations that were an essential part of a procedures-focused curriculum for decades can now be accomplished quickly and effectively using technology, so that students can now solve problems that were previously too time-consuming to attempt, and can focus on underlying concepts. “In an effective mathematics program, students learn in the presence of technology. Technology should influence the mathematics content taught and how it is taught. Powerful assistive and enabling computer and handheld technologies should be used seamlessly in teaching, learning, and assessment.”<sup>2</sup> This curriculum integrates appropriate technologies into the learning and doing of mathematics, while recognizing the continuing importance of students’ mastering essential numeric and algebraic skills.

## ROLES AND RESPONSIBILITIES IN MATHEMATICS PROGRAMS

### Students

Students have many responsibilities with regard to their learning. Students who make the effort required to succeed in school and who are able to apply themselves will soon discover that there is a direct relationship between this effort and their achievement, and will therefore be more motivated to work. There will be some students, however, who will find it more difficult to take responsibility for their learning because of special challenges they face. The attention, patience, and encouragement of teachers and family can be extremely important to these students’ success. However, taking responsibility for their own progress and learning is an important part of education for all students, regardless of their circumstances.

Mastery of concepts and skills in mathematics requires a sincere commitment to work and study. Students are expected to develop strategies and processes that facilitate learning and understanding in mathematics. Students should also be encouraged to actively pursue opportunities to apply their problem-solving skills outside the classroom and to extend and enrich their understanding of mathematics.

### Parents

Parents<sup>3</sup> have an important role to play in supporting student learning. Studies show that students perform better in school if their parents are involved in their education. By becoming familiar with the curriculum, parents can find out what is being taught in the courses their children are taking and what their children are expected to learn. This awareness will enhance parents’ ability to discuss their children’s work with them, to communicate with teachers, and to ask relevant questions about their children’s progress.

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2. Expert Panel on Student Success in Ontario, *Leading Math Success: Mathematical Literacy, Grades 7–12 – The Report of the Expert Panel on Student Success in Ontario, 2004* (Toronto: Ontario Ministry of Education, 2004), p. 47. (Referred to hereafter as *Leading Math Success*.)

3. The word *parents* is used throughout this document to stand for parent(s) and guardian(s).

Knowledge of the expectations in the various courses also helps parents to interpret teachers' comments on student progress and to work with them to improve student learning.

Effective ways for parents to support their children's learning include attending parent-teacher interviews, participating in parent workshops, becoming involved in school council activities (including becoming a school council member), and encouraging their children to complete their assignments at home.

The mathematics curriculum promotes lifelong learning. In addition to supporting regular school activities, parents can encourage their children to apply their problem-solving skills to other disciplines and to real-world situations.

## Teachers

Teachers and students have complementary responsibilities. Teachers are responsible for developing appropriate instructional strategies to help students achieve the curriculum expectations for their courses, as well as for developing appropriate methods for assessing and evaluating student learning. Teachers also support students in developing the reading, writing, and oral communication skills needed for success in their mathematics courses. Teachers bring enthusiasm and varied teaching and assessment approaches to the classroom, addressing different student needs and ensuring sound learning opportunities for every student.

Recognizing that students need a solid conceptual foundation in mathematics in order to further develop and apply their knowledge effectively, teachers endeavour to create a classroom environment that engages students' interest and helps them arrive at the understanding of mathematics that is critical to further learning.

Using a variety of instructional, assessment, and evaluation strategies, teachers provide numerous opportunities for students to develop skills of inquiry, problem solving, and communication as they investigate and learn fundamental concepts. The activities offered should enable students not only to make connections among these concepts throughout the course but also to relate and apply them to relevant societal, environmental, and economic contexts. Opportunities to relate knowledge and skills to these wider contexts – to the goals and concerns of the world in which they live – will motivate students to learn and to become lifelong learners.

## Principals

The principal works in partnership with teachers and parents to ensure that each student has access to the best possible educational experience. To support student learning, principals ensure that the Ontario curriculum is being properly implemented in all classrooms through the use of a variety of instructional approaches. They also ensure that appropriate resources are made available for teachers and students. To enhance teaching and learning in all subjects, including mathematics, principals promote learning teams and work with teachers to facilitate participation in professional-development activities.

Principals are also responsible for ensuring that every student who has an Individual Education Plan (IEP) is receiving the modifications and/or accommodations described in his or her plan – in other words, for ensuring that the IEP is properly developed, implemented, and monitored.

# THE PROGRAM IN MATHEMATICS

## OVERVIEW OF THE PROGRAM

The senior mathematics courses build on the Grade 9 and 10 program, relying on the same fundamental principles on which that program was based. Both are founded on the premise that students learn mathematics most effectively when they build a thorough understanding of mathematical concepts and procedures. Such understanding is achieved when mathematical concepts and procedures are introduced through an investigative approach and connected to students' prior knowledge in meaningful ways. This curriculum is designed to help students prepare for university, college, or the workplace by building a solid conceptual foundation in mathematics that will enable them to apply their knowledge and skills in a variety of ways and further their learning successfully.

An important part of every course in the mathematics program is the process of inquiry, in which students develop methods for exploring new problems or unfamiliar situations. Knowing how to learn mathematics is the underlying expectation that every student in every course needs to achieve. An important part of the inquiry process is that of taking the conditions of a real-world situation and representing them in mathematical form. A mathematical representation can take many different forms – for example, it can be a physical model, a diagram, a graph, a table of values, an equation, or a computer simulation. It is important that students recognize various mathematical representations of given relationships and that they become familiar with increasingly sophisticated representations as they progress through secondary school.

The prevalence in today's society and classrooms of sophisticated yet easy-to-use calculators and computer software accounts in part for the inclusion of certain concepts and skills in this curriculum. The curriculum has been designed to integrate appropriate technologies into the learning and doing of mathematics, while equipping students with the manipulation skills necessary to understand other aspects of the mathematics that they are learning, to solve meaningful problems, and to continue to learn mathematics with success in the future. Technology is not used to replace skill acquisition; rather, it is treated as a learning tool that helps students explore concepts. Technology is required when its use represents either the only way or the most effective way to achieve an expectation.

Like the earlier curriculum experienced by students, the senior secondary curriculum adopts a strong focus on the processes that best enable students to understand mathematical concepts and learn related skills. Attention to the mathematical processes is

considered to be essential to a balanced mathematics program. The seven mathematical processes identified in this curriculum are *problem solving, reasoning and proving, reflecting, selecting tools and computational strategies, connecting, representing, and communicating*. Each of the senior mathematics courses includes a set of expectations – referred to in this document as the “mathematical process expectations” – that outline the knowledge and skills involved in these essential processes. The mathematical processes apply to student learning in all areas of a mathematics course.

A balanced mathematics program at the secondary level also includes the development of algebraic skills. This curriculum has been designed to equip students with the algebraic skills needed to solve meaningful problems, to understand the mathematical concepts they are learning, and to successfully continue their study of mathematics in the future. The algebraic skills required in each course have been carefully chosen to support the topics included in the course. Calculators and other appropriate technologies will be used when the primary purpose of a given activity is the development of concepts or the solving of problems, or when situations arise in which computation or symbolic manipulation is of secondary importance.

### Courses in Grade 11 and Grade 12

Four types of courses are offered in the senior mathematics program: *university preparation, university/college preparation, college preparation, and workplace preparation*. Students choose course types on the basis of their interests, achievement, and postsecondary goals. The course types are defined as follows:

- *University preparation courses are designed to equip students with the knowledge and skills they need to meet the entrance requirements for university programs.*
- *University/college preparation courses are designed to equip students with the knowledge and skills they need to meet the entrance requirements for specific programs offered at universities and colleges.*
- *College preparation courses are designed to equip students with the knowledge and skills they need to meet the requirements for entrance to most college programs or for admission to specific apprenticeship or other training programs.*
- *Workplace preparation courses are designed to equip students with the knowledge and skills they need to meet the expectations of employers, if they plan to enter the workplace directly after graduation, or the requirements for admission to many apprenticeship or other training programs.*

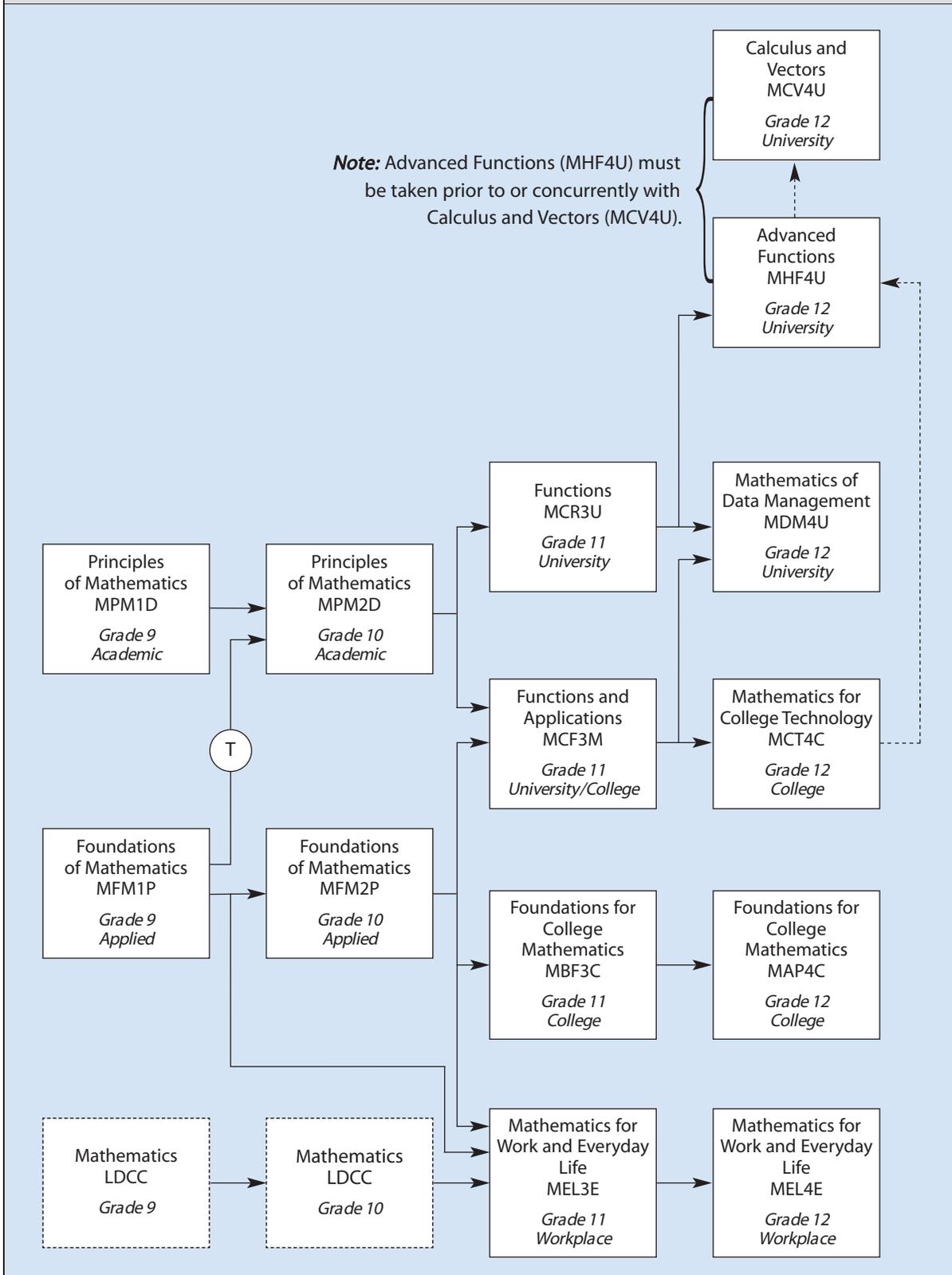
## Courses in Mathematics, Grades 11 and 12

Grade	Course Name	Course Type	Course Code	Prerequisite
11	Functions	University	MCR3U	Grade 10 Principles of Mathematics, Academic
11	Functions and Applications	University/College	MCF3M	Grade 10 Principles of Mathematics, Academic, or Grade 10 Foundations of Mathematics, Applied
11	Foundations for College Mathematics	College	MBF3C	Grade 10 Foundations of Mathematics, Applied Mathematics
11	Mathematics for Work and Everyday Life	Workplace	MEL3E	Grade 9 Principles of Mathematics, Academic, or Grade 9 Foundations of Mathematics, Applied, or a Grade 10 Mathematics LDCC (locally developed compulsory credit) course
12	Advanced Functions	University	MHF4U	Grade 11 Functions, University
12	Calculus and Vectors	University	MCV4U	Grade 12 Advanced Functions, University, must be taken prior to or concurrently with Calculus and Vectors.
12	Mathematics of Data Management	University	MDM4U	Grade 11 Functions, University, or Grade 11 Functions and Applications, University/College
12	Mathematics for College Technology	College	MCT4C	Grade 11 Functions and Applications, University/College, or Grade 11 Functions, University
12	Foundations for College Mathematics	College	MAP4C	Grade 11 Foundations for College Mathematics, College, or Grade 11 Functions and Applications, University/College
12	Mathematics for Work and Everyday Life	Workplace	MEL4E	Grade 11 Mathematics for Work and Everyday Life, Workplace

*Note:* Each of the courses listed above is worth one credit.

## Prerequisite Chart for Mathematics, Grades 9–12

This chart maps out all the courses in the discipline and shows the links between courses and the possible prerequisites for them. It does not attempt to depict all possible movements from course to course.



Notes:

- T – transfer course
- LDCC – locally developed compulsory credit course (LDCC courses are not outlined in this document.)

## Half-Credit Courses

The courses outlined in this document are designed to be offered as full-credit courses. However, *with the exception of the Grade 12 university preparation courses*, they may also be delivered as half-credit courses.

Half-credit courses, which require a minimum of fifty-five hours of scheduled instructional time, must adhere to the following conditions:

- The two half-credit courses created from a full course must together contain all of the expectations of the full course. The expectations for each half-credit course must be divided in a manner that best enables students to achieve the required knowledge and skills in the allotted time.
- A course that is a prerequisite for another course in the secondary curriculum may be offered as two half-credit courses, but students must successfully complete both parts of the course to fulfil the prerequisite. (Students are not required to complete both parts unless the course is a prerequisite for another course they wish to take.)
- The title of each half-credit course must include the designation *Part 1* or *Part 2*. A half credit (0.5) will be recorded in the credit-value column of both the report card and the Ontario Student Transcript.

Boards will ensure that all half-credit courses comply with the conditions described above, and will report all half-credit courses to the ministry annually in the School October Report.

## CURRICULUM EXPECTATIONS

The expectations identified for each course describe the knowledge and skills that students are expected to acquire, demonstrate, and apply in their class work, on tests, and in various other activities on which their achievement is assessed and evaluated.

Two sets of expectations are listed for each *strand*, or broad curriculum area, of each course.

- The *overall expectations* describe in general terms the knowledge and skills that students are expected to demonstrate by the end of each course.
- The *specific expectations* describe the expected knowledge and skills in greater detail. The specific expectations are arranged under numbered subheadings that relate to the overall expectations and that may serve as a guide for teachers as they plan learning activities for their students. The specific expectations are also numbered to indicate the overall expectation to which they relate (e.g., specific expectation 3.2 is related to overall expectation 3 in a given strand). The organization of expectations in subgroupings is not meant to imply that the expectations in any subgroup are achieved independently of the expectations in the other subgroups. The subheadings are used merely to help teachers focus on particular aspects of knowledge and skills as they develop and use various lessons and learning activities with their students.

In addition to the expectations outlined within each strand, a list of seven “mathematical process expectations” precedes the strands in all mathematics courses. These specific expectations describe the knowledge and skills that constitute processes essential to the effective study of mathematics. These processes apply to all areas of course content, and

students' proficiency in applying them must be developed in all strands of a mathematics course. Teachers should ensure that students develop their ability to apply these processes in appropriate ways as they work towards meeting the expectations outlined in the strands.

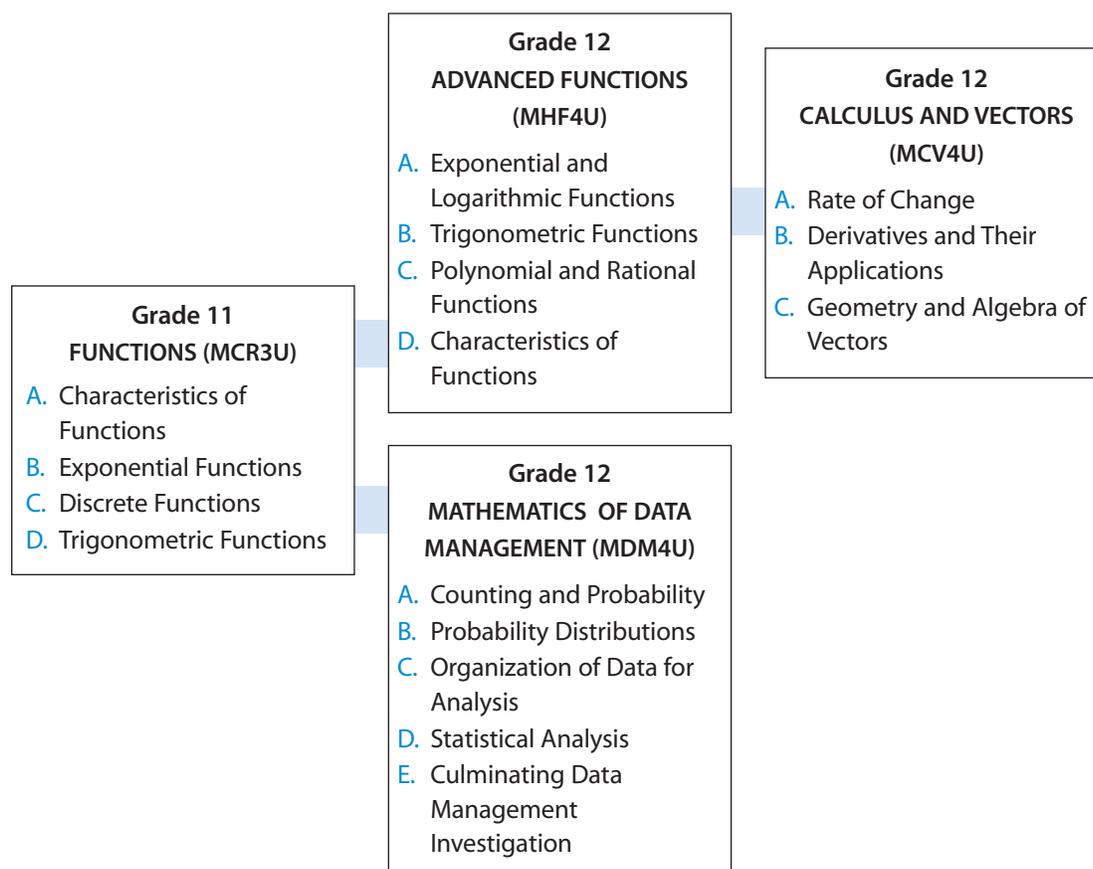
When developing detailed courses of study from this document, teachers are expected to weave together related expectations from different strands, as well as the relevant process expectations, in order to create an overall program that integrates and balances concept development, skill acquisition, the use of processes, and applications.

Many of the specific expectations are accompanied by examples and/or sample problems. These examples and sample problems are meant to illustrate the kind of skill, the specific area of learning, the depth of learning, and/or the level of complexity that the expectation entails. Some examples and sample problems may also be used to emphasize the importance of diversity or multiple perspectives. The examples and sample problems are intended only as suggestions for teachers. Teachers may incorporate the examples and sample problems into their lessons, or they may choose other topics, approaches, or problems that are relevant to the expectation.

## COURSES AND STRANDS

The courses in the Grade 11–12 mathematics curriculum are briefly described below, by course type. The strands in each course are listed in the graphic provided in each section, and their focus is discussed in the following text.

### University Preparation Courses



The **Grade 11 university preparation course, *Functions***, builds on the concepts and skills developed in the Grade 9 and 10 academic mathematics courses. The course is designed to prepare students for Grade 12 mathematics courses that lead to one of many university programs, including science, engineering, social sciences, liberal arts, and education. The concept of functions is introduced in the Characteristics of Functions strand of this course and extended through the investigation of two new types of relationships in the Exponential Functions and Trigonometric Functions strands. The Discrete Functions strand allows students, through the study of different representations of sequences and series, to revisit patterning and algebra concepts introduced in elementary school and make connections to financial applications involving compound interest and ordinary simple annuities.

The **Grade 12 university preparation course *Advanced Functions*** satisfies the mathematical prerequisite for some universities in areas that include business, social science, and health science programs. The strands in this course help students deepen their understanding of functions by revisiting the exponential and trigonometric functions introduced in Grade 11 to address related concepts such as radian measure and logarithmic functions and by extending prior knowledge of quadratic functions to explore polynomial and rational functions. The Characteristics of Functions strand addresses some of the general features of functions through the examination of rates of change and methods of combining functions.

The **Grade 12 university preparation course *Calculus and Vectors*** is designed to prepare students for university programs, such as science, engineering, and economics, that include a calculus or linear algebra course in the first year. Calculus is introduced in the Rate of Change strand by extending the numeric and graphical representation of rates of change introduced in the Advanced Functions course to include more abstract algebraic representations. The Derivatives and Their Applications strand provides students with the opportunity to develop the algebraic and problem-solving skills needed to solve problems associated with rates of change. Prior knowledge of geometry and trigonometry is used in the Geometry and Algebra of Vectors strand to develop vector concepts that can be used to solve interesting problems, including those arising from real-world applications.

The **Grade 12 university preparation course *Mathematics of Data Management*** is designed to satisfy the prerequisites for a number of university programs that may include statistics courses, such as those found in the social sciences and the humanities. The expectations in the strands of this course require students to apply mathematical process skills developed in prerequisite courses, such as problem solving, reasoning, and communication, to the study of probability and statistics. The Counting and Probability strand extends the basic probability concepts learned in the elementary school program and introduces counting techniques such as the use of permutations and combinations; these techniques are applied to both counting and probability problems. The Probability Distributions strand introduces the concept of probability distributions; these include the normal distribution, which is important in the study of statistics. In the Organization of Data for Analysis strand, students examine, use, and develop methods for organizing large amounts of data, while in the Statistical Analysis strand, students investigate and develop an understanding of powerful concepts used to analyse and interpret large amounts of data. These concepts are developed with the use of technological tools such

as spreadsheets and *Fathom*, a ministry-licensed dynamic statistical program. The Culminating Data Management Investigation strand requires students to undertake a culminating investigation dealing with a significant issue that will require the application of the skills from the other strands of the course.

## University/College Preparation and College Preparation Courses

<p><b>Grade 11</b> <b>FUNCTIONS AND APPLICATIONS (MCF3M)</b></p> <ul style="list-style-type: none"><li>A. Quadratic Functions</li><li>B. Exponential Functions</li><li>C. Trigonometric Functions</li></ul>	<p><b>Grade 12</b> <b>MATHEMATICS FOR COLLEGE TECHNOLOGY (MCT4C)</b></p> <ul style="list-style-type: none"><li>A. Exponential Functions</li><li>B. Polynomial Functions</li><li>C. Trigonometric Functions</li><li>D. Applications of Geometry</li></ul>
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The **Grade 11 university/college preparation course**, *Functions and Applications*, provides preparation for students who plan to pursue technology-related programs in college, while also leaving the option open for some students to pursue postsecondary programs that require the Grade 12 university preparation course Mathematics of Data Management. The Functions and Applications course explores functions by revisiting key concepts from the Grade 10 mathematics curriculum and by using a more applied approach with less emphasis on abstract concepts than in the Grade 11 university preparation course, Functions. The first strand, Quadratic Functions, extends knowledge and skills related to quadratics for students who completed the Grade 10 applied mathematics course and reviews this topic for students entering from the Grade 10 academic course. The strand also introduces some of the properties of functions. The other two strands, Exponential Functions and Trigonometric Functions, emphasize real-world applications and help students develop the knowledge and skills needed to solve problems related to these applications.

The **Grade 12 college preparation course** *Mathematics for College Technology* provides excellent preparation for success in technology-related programs at the college level. It extends the understanding of functions developed in the Grade 11 university/college preparation course, Functions and Applications, using a more applied approach, and may help students who decide to pursue certain university programs to prepare for the Grade 12 university preparation course Advanced Functions. Exponential and trigonometric functions are revisited, developing algebraic skills needed to solve problems involving exponential equations and extending the skills associated with graphical representations of trigonometric functions. The Polynomial Functions strand extends to polynomial functions concepts that connect graphs and equations of quadratic functions. Finally, students apply geometric relationships to solve problems involving composite shapes and figures and investigate the properties of circles and their applications.

<b>Grade 11</b> <b>FOUNDATIONS FOR</b> <b>COLLEGE</b> <b>MATHEMATICS (MBF3C)</b>	<b>Grade 12</b> <b>FOUNDATIONS FOR</b> <b>COLLEGE</b> <b>MATHEMATICS (MAP4C)</b>
<ul style="list-style-type: none"> <li>A. Mathematical Models</li> <li>B. Personal Finance</li> <li>C. Geometry and Trigonometry</li> <li>D. Data Management</li> </ul>	<ul style="list-style-type: none"> <li>A. Mathematical Models</li> <li>B. Personal Finance</li> <li>C. Geometry and Trigonometry</li> <li>D. Data Management</li> </ul>

The **Grade 11 college preparation course**, *Foundations for College Mathematics*, includes a blend of topics needed by students who plan to pursue one of a broad range of college programs. The course has been designed with four strands that address different areas of mathematics. The Mathematical Models strand uses the concepts connected to linear and quadratic relations developed in the Grade 9 and 10 applied mathematics courses to revisit quadratic relations and introduce exponential relations. The Personal Finance strand focuses on compound interest and applications related to investing and borrowing money and owning and operating a vehicle. Applications requiring spatial reasoning are addressed in the Geometry and Trigonometry strand. The fourth strand, Data Management, explores practical applications of one-variable statistics and probability.

The **Grade 12 college preparation course** *Foundations for College Mathematics* satisfies the mathematical prerequisites for many college programs, including programs in business, human services, hospitality and tourism, and some of the health sciences. The four strands of this course focus on the same areas of mathematics addressed in the Grade 11 college preparation course, *Foundations for College Mathematics*. The Mathematical Models strand extends the concepts and skills that related to exponential relations introduced in Grade 11 and provides students with an opportunity to revisit all of the relations they have studied in the secondary mathematics program by using a graphical and algebraic approach. The Personal Finance strand focuses on annuities and mortgages, renting or owning accommodation, and designing budgets. Problem solving in the Geometry and Trigonometry strand reinforces the application of relationships associated with a variety of shapes and figures. The fourth strand, Data Management, addresses practical applications of two-variable statistics and examines applications of data management.

### Workplace Preparation Courses

<b>Grade 11</b> <b>MATHEMATICS FOR WORK</b> <b>AND EVERYDAY LIFE</b> <b>(MEL3E)</b>	<b>Grade 12</b> <b>MATHEMATICS FOR WORK</b> <b>AND EVERYDAY LIFE</b> <b>(MEL4E)</b>
<ul style="list-style-type: none"> <li>A. Earning and Purchasing</li> <li>B. Saving, Investing, and Borrowing</li> <li>C. Transportation and Travel</li> </ul>	<ul style="list-style-type: none"> <li>A. Reasoning With Data</li> <li>B. Personal Finance</li> <li>C. Applications of Measurement</li> </ul>

The **Grade 11 workplace preparation course, *Mathematics for Work and Everyday Life***, is designed to help students consolidate the basic knowledge and skills of mathematics used in the workplace and in everyday life. This course is ideal for students who would like to take the Grade 12 workplace preparation course before graduating from high school and entering the workplace. The course also meets the needs of students who wish to fulfill the senior mathematics graduation requirement but do not plan to take any further courses in mathematics. All three strands, Earning and Purchasing; Saving, Investing, and Borrowing; and Transportation and Travel, provide students with the opportunity to use proportional reasoning to solve a variety of problems.

The **Grade 12 workplace preparation course, *Mathematics for Work and Everyday Life***, extends the knowledge and skills developed in Grade 11. The gathering, interpretation, and display of one-variable data and the investigation of probability concepts are the main components of the Reasoning With Data strand. Topics in the Personal Finance strand address owning or renting accommodation, designing a budget, and filing an income tax return. A variety of problems involving metric and imperial measurement are presented in the Applications of Measurement strand. The expectations support the use of hands-on projects and other experiences that make the mathematics more meaningful for students.

# THE MATHEMATICAL PROCESSES

Presented at the start of every course in this curriculum document are seven mathematical process expectations that describe a set of skills that support lifelong learning in mathematics and that students need to develop on an ongoing basis, as they work to achieve the expectations outlined within each course. In the 2000 mathematics curriculum, expectations that addressed the mathematical processes were present within individual strands to varying degrees. Here, the mathematical processes are highlighted in each course to ensure that students are actively engaged in developing their skills to apply them throughout the course, rather than only in specific strands.

The mathematical processes are as follows:

- problem solving
- reasoning and proving
- reflecting
- selecting tools and computational strategies
- connecting
- representing
- communicating

Each course presents students with rich problem-solving experiences through a variety of approaches, including investigation. These experiences provide students with opportunities to develop and apply the mathematical processes.

The mathematical processes are interconnected. Problem solving and communicating have strong links to all the other processes. The problem-solving process can be thought of as the motor that drives the development of the other processes. It allows students to make conjectures and to reason as they pursue a solution or a new understanding. Problem solving provides students with the opportunity to make connections to their prior learning and to make decisions about the representations, tools, and computational strategies needed to solve the problem. Teachers should encourage students to justify their solutions, communicate them orally and in writing, and reflect on alternative solutions. By seeing how others solve a problem, students can begin to think about their own thinking (metacognition) and the thinking of others, and to consciously adjust their own strategies in order to make their solutions as efficient and accurate as possible.

The mathematical processes cannot be separated from the knowledge and skills that students acquire throughout the course. Students who problem solve, communicate, reason, reflect, and so on, as they learn mathematics, will develop the knowledge, the understanding of concepts, and the skills required in the course in a more meaningful way.

## PROBLEM SOLVING

Problem solving is central to learning mathematics. It forms the basis of effective mathematics programs and should be the mainstay of mathematical instruction. It is considered an essential process through which students are able to achieve the expectations in mathematics, and is an integral part of the mathematics curriculum in Ontario, for the following reasons. Problem solving:

- helps students become more confident mathematicians;
- allows students to use the knowledge they bring to school and helps them connect mathematics with situations outside the classroom;
- helps students develop mathematical understanding and gives meaning to skills and concepts in all strands;
- allows students to reason, communicate ideas, make connections, and apply knowledge and skills;
- offers excellent opportunities for assessing students' understanding of concepts, ability to solve problems, ability to apply concepts and procedures, and ability to communicate ideas;
- promotes collaborative sharing of ideas and strategies, and promotes talking about mathematics;
- helps students find enjoyment in mathematics;
- increases opportunities for the use of critical-thinking skills (e.g., estimating, classifying, assuming, recognizing relationships, hypothesizing, offering opinions with reasons, evaluating results, and making judgements).

Not all mathematics instruction, however, can take place in a problem-solving context. Certain aspects of mathematics must be explicitly taught. Conventions, including the use of mathematical symbols and terms, are one such aspect, and they should be introduced to students as needed, to enable them to use the symbolic language of mathematics.

### Selecting Problem-Solving Strategies

Problem-solving strategies are methods that can be used to solve various types of problems. Common problem-solving strategies include: making a model, picture, or diagram; looking for a pattern; guessing and checking; making assumptions; creating an organized list; making a table or chart; solving a simpler problem; working backwards; and using logical reasoning.

Teachers who use problem solving as a focus of their mathematics teaching help students develop and extend a repertoire of strategies and methods that they can apply when solving various kinds of problems – instructional problems, routine problems, and non-routine problems. Students develop this repertoire over time, as their problem-solving skills mature. By secondary school, students will have learned many problem-solving strategies that they can flexibly use to investigate mathematical concepts or can apply when faced with unfamiliar problem-solving situations.

## REASONING AND PROVING

Reasoning helps students make sense of mathematics. Classroom instruction in mathematics should foster critical thinking – that is, an organized, analytical, well-reasoned approach to learning mathematical concepts and processes and to solving problems.

As students investigate and make conjectures about mathematical concepts and relationships, they learn to employ *inductive reasoning*, making generalizations based on specific findings from their investigations. Students also learn to use counter-examples to disprove conjectures. Students can use *deductive reasoning* to assess the validity of conjectures and to formulate proofs.

## REFLECTING

Good problem-solvers regularly and consciously reflect on and monitor their own thought processes. By doing so, they are able to recognize when the technique they are using is not fruitful, and to make a conscious decision to switch to a different strategy, rethink the problem, search for related content knowledge that may be helpful, and so forth. Students' problem-solving skills are enhanced when they reflect on alternative ways to perform a task even if they have successfully completed it. Reflecting on the reasonableness of an answer by considering the original question or problem is another way in which students can improve their ability to make sense of problems.

## SELECTING TOOLS AND COMPUTATIONAL STRATEGIES

The primary role of learning tools such as calculators, manipulatives, graphing technologies, computer algebra systems, dynamic geometry software, and dynamic statistical software is to help students develop a deeper understanding of mathematics through the use of a variety of tools and strategies. Students need to develop the ability to select the appropriate learning tools and computational strategies to perform particular mathematical tasks, to investigate mathematical ideas, and to solve problems.

### Calculators, Computers, Communications Technology

Various types of technology are useful in learning and doing mathematics. Students can use calculators and computers to extend their capacity to investigate and analyse mathematical concepts and to reduce the time they might otherwise spend on purely mechanical activities.

Technology helps students perform operations, make graphs, manipulate algebraic expressions, and organize and display data that are lengthier or more complex than those addressed in curriculum expectations suited to a paper-and-pencil approach. It can be used to investigate number and graphing patterns, geometric relationships, and different representations; to simulate situations; and to extend problem solving. Students also need to recognize when it is appropriate to apply their mental computation, reasoning, and estimation skills to predict results and check answers.

Technologies must be seen as important problem-solving tools. Computers and calculators are tools of mathematicians, and students should be given opportunities to select and use the learning tools that may be helpful to them as they search for their own solutions to problems.

It is important that teachers introduce the use of technology in ways that build students' confidence and contribute to their understanding of the concepts being investigated, especially when students may not be familiar with the use of some of the technologies suggested in the curriculum. Students' use of technology should not be laborious or restricted to inputting and learning algorithmic steps. For example, when using spreadsheets and statistical software (e.g., *Fathom*), teachers could supply students with prepared data sets, and when using dynamic geometry software (e.g., *The Geometer's Sketchpad*), pre-made sketches could be used to ensure that students focus on the important mathematical relationships, and not just on the inputting of data or on the construction of the sketch.

Whenever appropriate, students should be encouraged to select and use the communications technology that would best support and communicate their learning. Computer software programs can help students collect, organize, and sort the data they gather, and write, edit, and present reports on their findings. Students, working individually or in groups, can use Internet websites to gain access to Statistics Canada, mathematics organizations, and other valuable sources of mathematical information around the world.

### Manipulatives

Although technologies are the most common learning tools used by students studying senior level mathematics, students should still be encouraged, when appropriate, to select and use concrete learning tools to make models of mathematical ideas. Students need to understand that making their own models is a powerful means of building understanding and explaining their thinking to others.

Representation of mathematical ideas using manipulatives<sup>4</sup> helps students to:

- see patterns and relationships;
- make connections between the concrete and the abstract;
- test, revise, and confirm their reasoning;
- remember how they solved a problem;
- communicate their reasoning to others.

### Computational Strategies

Problem solving often requires students to select an appropriate computational strategy such as applying a standard algorithm, using technology, or applying strategies related to mental computation and estimation. Developing the ability to perform mental computation and to estimate is an important aspect of student learning in mathematics. Knowing when to apply such skills is equally important.

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4. See the Instructional Approaches section, on page 30 of this document, for additional information about the use of manipulatives in mathematics instruction.

Mental computation involves calculations done in the mind, with little or no use of paper and pencil. Students who have developed the ability to calculate mentally can select from and use a variety of procedures that take advantage of their knowledge and understanding of numbers, the operations, and their properties. Using knowledge of the distributive property, for example, students can mentally compute 70% of 22 by first considering 70% of 20 and then adding 70% of 2. Used effectively, mental computation can encourage students to think more deeply about numbers and number relationships.

Knowing how to estimate and recognizing when it is useful to estimate and when it is necessary to have an exact answer are important mathematical skills. Estimation is a useful tool for judging the reasonableness of a solution and for guiding students in their use of calculators. The ability to estimate depends on a well-developed sense of number and an understanding of place value. It can be a complex skill that requires decomposing numbers, compensating for errors, and perhaps even restructuring the problem. Estimation should not be taught as an isolated skill or a set of isolated rules and techniques. Recognizing calculations that are easy to perform and developing fluency in performing basic operations contribute to successful estimation.

## CONNECTING

Experiences that allow students to make more connections – to see, for example, how concepts and skills from one strand of mathematics are related to those from another or how a mathematical concept can be applied in the real world – will help them develop deeper mathematical understanding. As they continue to make such connections, students begin to see mathematics more as a study of relationships rather than a series of isolated skills and concepts. Making connections not only deepens understanding, but also helps students develop the ability to use learning from one area of mathematics to understand another.

Making connections between the mathematics being studied and its applications in the real world helps convince students of the usefulness and relevance of mathematics beyond the classroom.

## REPRESENTING

In the senior mathematics curriculum, representing mathematical ideas and modelling situations generally involve concrete, numeric, graphical, and algebraic representations. Pictorial, geometric representations as well as representations using dynamic software can also be very helpful. Students should be able to recognize the connections between representations, translate one representation into another, and use the different representations appropriately and as needed to solve problems. Knowing the different ways in which a mathematical idea can be represented helps students develop a better understanding of mathematical concepts and relationships; communicate their thinking and understanding; recognize connections among related mathematical concepts; and model and interpret mathematical, physical, and social phenomena. When students are able to represent concepts in various ways, they develop flexibility in their thinking about those concepts. They are not inclined to perceive any single representation as “the math”; rather, they understand that it is just one of many representations that help them understand a concept.

## COMMUNICATING

Communication is the process of expressing mathematical ideas and understandings orally, visually, and in writing, using numbers, symbols, pictures, graphs, diagrams, and words. Providing effective explanations and using correct mathematical notation when developing and presenting mathematical ideas and solutions are key aspects of effective communication in mathematics. Students communicate for various purposes and for different audiences, such as the teacher, a peer, a group of students, or the whole class. Communication is an essential process in learning mathematics. Through communication, students are able to reflect upon and clarify ideas, relationships, and mathematical arguments.

Many opportunities exist for teachers to help students develop their ability to communicate mathematically. For example, teachers can:

- model proper use of symbols, vocabulary, and notations in oral and written form;
- expect correct use of mathematical symbols and conventions in student work;
- ensure that students are exposed to and use new mathematical vocabulary as it is introduced (e.g., as they gather and interpret information; by providing opportunities to read, question, and discuss);
- provide feedback to students on their use of terminology and conventions;
- ask clarifying and extending questions and encourage students to ask themselves similar kinds of questions;
- ask students open-ended questions relating to specific topics or information;
- model ways in which various kinds of questions can be answered.

Effective classroom communication requires a supportive and respectful environment that makes all members of the class comfortable when they speak and when they question, react to, and elaborate on the statements of their classmates and the teacher.

# ASSESSMENT AND EVALUATION OF STUDENT ACHIEVEMENT

## **BASIC CONSIDERATIONS**

The primary purpose of assessment and evaluation is to improve student learning. Information gathered through assessment helps teachers to determine students' strengths and weaknesses in their achievement of the curriculum expectations in each course. This information also serves to guide teachers in adapting curriculum and instructional approaches to students' needs and in assessing the overall effectiveness of programs and classroom practices.

Assessment is the process of gathering information from a variety of sources (including assignments, demonstrations, projects, performances, and tests) that accurately reflects how well a student is achieving the curriculum expectations in a course. As part of assessment, teachers provide students with descriptive feedback that guides their efforts towards improvement. Evaluation refers to the process of judging the quality of student work on the basis of established criteria, and assigning a value to represent that quality.

Assessment and evaluation will be based on the provincial curriculum expectations and the achievement levels outlined in this document.

In order to ensure that assessment and evaluation are valid and reliable, and that they lead to the improvement of student learning, teachers must use assessment and evaluation strategies that:

- address both what students learn and how well they learn;
- are based both on the categories of knowledge and skills and on the achievement level descriptions given in the achievement chart on pages 28–29;
- are varied in nature, administered over a period of time, and designed to provide opportunities for students to demonstrate the full range of their learning;
- are appropriate for the learning activities used, the purposes of instruction, and the needs and experiences of the students;

- are fair to all students;
- accommodate students with special education needs, consistent with the strategies outlined in their Individual Education Plan;
- accommodate the needs of students who are learning the language of instruction (English or French);
- ensure that each student is given clear directions for improvement;
- promote students' ability to assess their own learning and to set specific goals;
- include the use of samples that provide evidence of their achievement;
- are communicated clearly to students and parents at the beginning of the course or the school term and at other appropriate points throughout the school year.

All curriculum expectations must be accounted for in instruction, but evaluation focuses on students' achievement of the overall expectations. A student's achievement of the overall expectations is evaluated on the basis of his or her achievement of related specific expectations (including the process expectations). The overall expectations are broad in nature, and the specific expectations define the particular content or scope of the knowledge and skills referred to in the overall expectations. Teachers will use their professional judgement to determine which specific expectations should be used to evaluate achievement of the overall expectations, and which ones will be covered in instruction and assessment (e.g., through direct observation) but not necessarily evaluated.

The characteristics given in the achievement chart (pages 28–29) for level 3 represent the “provincial standard” for achievement of the expectations in a course. A complete picture of overall achievement at level 3 in a course in mathematics can be constructed by reading from top to bottom in the shaded column of the achievement chart, headed “70–79% (Level 3)”. Parents of students achieving at level 3 can be confident that their children will be prepared for work in subsequent courses.

Level 1 identifies achievement that falls much below the provincial standard, while still reflecting a passing grade. Level 2 identifies achievement that approaches the standard. Level 4 identifies achievement that surpasses the standard. It should be noted that achievement at level 4 does not mean that the student has achieved expectations beyond those specified for a particular course. It indicates that the student has achieved all or almost all of the expectations for that course, and that he or she demonstrates the ability to use the specified knowledge and skills in more sophisticated ways than a student achieving at level 3.

## THE ACHIEVEMENT CHART FOR MATHEMATICS

The achievement chart for mathematics (see pages 28–29) identifies four categories of knowledge and skills. The achievement chart is a standard province-wide guide to be used by teachers. It enables teachers to make judgements about student work that are based on clear performance standards and on a body of evidence collected over time.

The purpose of the achievement chart is to:

- provide a common framework that encompasses the curriculum expectations for all courses outlined in this document;
- guide the development of quality assessment tasks and tools (including rubrics);
- help teachers to plan instruction for learning;
- assist teachers in providing meaningful feedback to students;
- provide various categories and criteria with which to assess and evaluate student learning.

### Categories of Knowledge and Skills

The categories, defined by clear criteria, represent four broad areas of knowledge and skills within which the expectations for any given mathematics course are organized. The four categories should be considered as interrelated, reflecting the wholeness and interconnectedness of learning.

The categories of knowledge and skills are described as follows:

**Knowledge and Understanding.** Subject-specific content acquired in each course (knowledge), and the comprehension of its meaning and significance (understanding).

**Thinking.** The use of critical and creative thinking skills and/or processes,<sup>5</sup> as follows:

- planning skills (e.g., understanding the problem, making a plan for solving the problem)
- processing skills (e.g., carrying out a plan, looking back at the solution)
- critical/creative thinking processes (e.g., inquiry, problem solving)

**Communication.** The conveying of meaning through various oral, written, and visual forms (e.g., providing explanations of reasoning or justification of results orally or in writing; communicating mathematical ideas and solutions in writing, using numbers and algebraic symbols, and visually, using pictures, diagrams, charts, tables, graphs, and concrete materials).

**Application.** The use of knowledge and skills to make connections within and between various contexts.

Teachers will ensure that student work is assessed and/or evaluated in a balanced manner with respect to the four categories, and that achievement of particular expectations is considered within the appropriate categories.

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5. See the footnote on page 28, pertaining to the mathematical processes.

## Criteria

Within each category in the achievement chart, criteria are provided that are subsets of the knowledge and skills that define each category. For example, in Knowledge and Understanding, the criteria are “knowledge of content (e.g., facts, terms, procedural skills, use of tools)” and “understanding of mathematical concepts”. The criteria identify the aspects of student performance that are assessed and/or evaluated, and serve as guides to what to look for.

## Descriptors

A “descriptor” indicates the characteristic of the student’s performance, with respect to a particular criterion, on which assessment or evaluation is focused. In the achievement chart, *effectiveness* is the descriptor used for each criterion in the Thinking, Communication, and Application categories. What constitutes effectiveness in any given performance task will vary with the particular criterion being considered. Assessment of effectiveness may therefore focus on a quality such as appropriateness, clarity, accuracy, precision, logic, relevance, significance, fluency, flexibility, depth, or breadth, as appropriate for the particular criterion. For example, in the Thinking category, assessment of effectiveness might focus on the degree of relevance or depth apparent in an analysis; in the Communication category, on clarity of expression or logical organization of information and ideas; or in the Application category, on appropriateness or breadth in the making of connections. Similarly, in the Knowledge and Understanding category, assessment of knowledge might focus on accuracy, and assessment of understanding might focus on the depth of an explanation. Descriptors help teachers to focus their assessment and evaluation on specific knowledge and skills for each category and criterion, and help students to better understand exactly what is being assessed and evaluated.

## Qualifiers

A specific “qualifier” is used to define each of the four levels of achievement – that is, *limited* for level 1, *some* for level 2, *considerable* for level 3, and a *high degree* or *thorough* for level 4. A qualifier is used along with a descriptor to produce a description of performance at a particular level. For example, the description of a student’s performance at level 3 with respect to the first criterion in the Thinking category would be: “the student uses planning skills with *considerable* effectiveness”.

The descriptions of the levels of achievement given in the chart should be used to identify the level at which the student has achieved the expectations. In all of their courses, students should be provided with numerous and varied opportunities to demonstrate the full extent of their achievement of the curriculum expectations, across all four categories of knowledge and skills.

## EVALUATION AND REPORTING OF STUDENT ACHIEVEMENT

Student achievement must be communicated formally to students and parents by means of the Provincial Report Card, Grades 9–12. The report card provides a record of the student’s achievement of the curriculum expectations in every course, at particular points in the school year or semester, in the form of a percentage grade. The percentage grade represents the quality of the student’s overall achievement of the expectations for the

course and reflects the corresponding level of achievement as described in the achievement chart for the discipline.

A final grade is recorded for every course, and a credit is granted and recorded for every course in which the student's grade is 50% or higher. The final grade for each course in Grades 9–12 will be determined as follows:

- Seventy per cent of the grade will be based on evaluations conducted throughout the course. This portion of the grade should reflect the student's most consistent level of achievement throughout the course, although special consideration should be given to more recent evidence of achievement.
- Thirty per cent of the grade will be based on a final evaluation in the form of an examination, performance, essay, and/or other method of evaluation suitable to the course content and administered towards the end of the course.

## **REPORTING ON DEMONSTRATED LEARNING SKILLS**

The report card provides a record of the learning skills demonstrated by the student in every course, in the following five categories: Works Independently, Teamwork, Organization, Work Habits, and Initiative. The learning skills are evaluated using a four-point scale (E-Excellent, G-Good, S-Satisfactory, N-Needs Improvement). The separate evaluation and reporting of the learning skills in these five areas reflect their critical role in students' achievement of the curriculum expectations. To the extent possible, the evaluation of learning skills, apart from any that may be included as part of a curriculum expectation in a course, should not be considered in the determination of percentage grades.

## ACHIEVEMENT CHART: MATHEMATICS, GRADES 9–12

Categories	50–59% (Level 1)	60–69% (Level 2)	70–79% (Level 3)	80–100% (Level 4)
<b>Knowledge and Understanding</b> – Subject-specific content acquired in each course (knowledge), and the comprehension of its meaning and significance (understanding)				
	The student:			
<b>Knowledge of content</b> (e.g., facts, terms, procedural skills, use of tools)	demonstrates limited knowledge of content	demonstrates some knowledge of content	demonstrates considerable knowledge of content	demonstrates thorough knowledge of content
<b>Understanding of mathematical concepts</b>	demonstrates limited understanding of concepts	demonstrates some understanding of concepts	demonstrates considerable understanding of concepts	demonstrates thorough understanding of concepts
<b>Thinking</b> – The use of critical and creative thinking skills and/or processes*				
	The student:			
<b>Use of planning skills</b> – understanding the problem (e.g., formulating and interpreting the problem, making conjectures) – making a plan for solving the problem	uses planning skills with limited effectiveness	uses planning skills with some effectiveness	uses planning skills with considerable effectiveness	uses planning skills with a high degree of effectiveness
<b>Use of processing skills</b> – carrying out a plan (e.g., collecting data, questioning, testing, revising, modelling, solving, inferring, forming conclusions) – looking back at the solution (e.g., evaluating reasonableness, making convincing arguments, reasoning, justifying, proving, reflecting)	uses processing skills with limited effectiveness	uses processing skills with some effectiveness	uses processing skills with considerable effectiveness	uses processing skills with a high degree of effectiveness
<b>Use of critical/creative thinking processes</b> (e.g., problem solving, inquiry)	uses critical/creative thinking processes with limited effectiveness	uses critical/creative thinking processes with some effectiveness	uses critical/creative thinking processes with considerable effectiveness	uses critical/creative thinking processes with a high degree of effectiveness

\* The processing skills and critical/creative thinking processes in the Thinking category include some but not all aspects of the mathematical processes described on pages 17–22 of this document. Some aspects of the mathematical processes relate to the other categories of the achievement chart.

Categories	50–59% (Level 1)	60–69% (Level 2)	70–79% (Level 3)	80–100% (Level 4)
<b>Communication</b> – The conveying of meaning through various forms				
	The student:			
<b>Expression and organization of ideas and mathematical thinking</b> (e.g., clarity of expression, logical organization), <b>using oral, visual, and written forms</b> (e.g., pictorial, graphic, dynamic, numeric, algebraic forms; concrete materials)	expresses and organizes mathematical thinking with limited effectiveness	expresses and organizes mathematical thinking with some effectiveness	expresses and organizes mathematical thinking with considerable effectiveness	expresses and organizes mathematical thinking with a high degree of effectiveness
<b>Communication for different audiences and purposes</b> (e.g., peers, teachers) and <b>purposes</b> (e.g., to present data, justify a solution, express a mathematical argument) <b>in oral, visual, and written forms</b>	communicates for different audiences and purposes with limited effectiveness	communicates for different audiences and purposes with some effectiveness	communicates for different audiences and purposes with considerable effectiveness	communicates for different audiences and purposes with a high degree of effectiveness
<b>Use of conventions, vocabulary, and terminology of the discipline</b> (e.g., terms, symbols) <b>in oral, visual, and written forms</b>	uses conventions, vocabulary, and terminology of the discipline with limited effectiveness	uses conventions, vocabulary, and terminology of the discipline with some effectiveness	uses conventions, vocabulary, and terminology of the discipline with considerable effectiveness	uses conventions, vocabulary, and terminology of the discipline with a high degree of effectiveness
<b>Application</b> – The use of knowledge and skills to make connections within and between various contexts				
	The student:			
<b>Application of knowledge and skills in familiar contexts</b>	applies knowledge and skills in familiar contexts with limited effectiveness	applies knowledge and skills in familiar contexts with some effectiveness	applies knowledge and skills in familiar contexts with considerable effectiveness	applies knowledge and skills in familiar contexts with a high degree of effectiveness
<b>Transfer of knowledge and skills to new contexts</b>	transfers knowledge and skills to new contexts with limited effectiveness	transfers knowledge and skills to new contexts with some effectiveness	transfers knowledge and skills to new contexts with considerable effectiveness	transfers knowledge and skills to new contexts with a high degree of effectiveness
<b>Making connections within and between various contexts</b> (e.g., connections between concepts, representations, and forms within mathematics; connections involving use of prior knowledge and experience; connections between mathematics, other disciplines, and the real world)	makes connections within and between various contexts with limited effectiveness	makes connections within and between various contexts with some effectiveness	makes connections within and between various contexts with considerable effectiveness	makes connections within and between various contexts with a high degree of effectiveness

Note: A student whose achievement is below 50% at the end of a course will not obtain a credit for the course.

# SOME CONSIDERATIONS FOR PROGRAM PLANNING IN MATHEMATICS

Teachers who are planning a program in mathematics must take into account considerations in a number of important areas, including those discussed below.

## **INSTRUCTIONAL APPROACHES**

To make new learning more accessible to students, teachers build new learning upon the knowledge and skills students have acquired in previous years – in other words, they help activate prior knowledge. It is important to assess where students are in their mathematical growth and to bring them forward in their learning.

In order to apply their knowledge effectively and to continue to learn, students must have a solid conceptual foundation in mathematics. Successful classroom practices engage students in activities that require higher-order thinking, with an emphasis on problem solving.<sup>6</sup> Learning experienced in the primary, junior, and intermediate divisions should have provided students with a good grounding in the investigative approach to learning new mathematical concepts, including inquiry models of problem solving, and this approach continues to be important in the senior mathematics program.

Students in a mathematics class typically demonstrate diversity in the ways they learn best. It is important, therefore, that students have opportunities to learn in a variety of ways – individually, cooperatively, independently, with teacher direction, through investigation involving hands-on experience, and through examples followed by practice. In mathematics, students are required to learn concepts, acquire procedures and skills, and apply processes with the aid of the instructional and learning strategies best suited to the particular type of learning.

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6. See the resource document *Targeted Implementation & Planning Supports for Revised Mathematics (TIPS4RM): Grade 7, 8, 9 Applied and 10 Applied* (Toronto: Queen's Printer for Ontario, 2005) for helpful information about problem solving.

The approaches and strategies used in the classroom to help students meet the expectations of this curriculum will vary according to the object of the learning and the needs of the students. For example, even at the secondary level, manipulatives can be important tools for supporting the effective learning of mathematics. These concrete learning tools, such as connecting cubes, measurement tools, algebra tiles, and number cubes, invite students to explore and represent abstract mathematical ideas in varied, concrete, tactile, and visually rich ways.<sup>7</sup> Other representations, including graphical and algebraic representations, are also a valuable aid to teachers. By analysing students' representations of mathematical concepts and listening carefully to their reasoning, teachers can gain useful insights into students' thinking and provide supports to help enhance their thinking.

All learning, especially new learning, should be embedded in well-chosen contexts for learning – that is, contexts that are broad enough to allow students to investigate initial understandings, identify and develop relevant supporting skills, and gain experience with varied and interesting applications of the new knowledge. Such rich contexts for learning open the door for students to see the “big ideas” of mathematics – that is, the major underlying principles or relationships that will enable and encourage students to reason mathematically throughout their lives.

### Promoting Positive Attitudes Towards Learning Mathematics

Students' attitudes have a significant effect on how students approach problem solving and how well they succeed in mathematics. Students who enjoy mathematics tend to perform well in their mathematics course work and are more likely to enrol in the more advanced mathematics courses.

Students develop positive attitudes when they are engaged in making mathematical conjectures, when they experience breakthroughs as they solve problems, when they see connections between important ideas, and when they observe an enthusiasm for mathematics on the part of their teachers.<sup>8</sup> With a positive attitude towards mathematics, students are able to make more sense of the mathematics they are working on, and to view themselves as effective learners of mathematics. They are also more likely to perceive mathematics as both useful and worthwhile, and to develop the belief that steady effort in learning mathematics pays off.

It is common for people to feel inadequate or anxious when they cannot solve problems quickly and easily, or in the right way. To gain confidence, students need to recognize that, for some mathematics problems, there may be several ways to arrive at a solution. They also need to understand that problem solving of almost any kind often requires a considerable expenditure of time and energy and a good deal of perseverance. To counteract the frustration they may feel when they are not making progress towards solving a problem, they need to believe that they are capable of finding solutions. Teachers can encourage students to develop a willingness to persist, to investigate, to reason, to explore alternative solutions, to view challenges as opportunities to extend their learning, and to take the risks necessary to become successful problem solvers. They can help students develop confidence and reduce anxiety and frustration by providing them with problems that are challenging but not beyond their ability to solve. Problems at a developmentally appropriate level help students to learn while establishing a norm of perseverance for successful problem solving.

7. A list of manipulatives appropriate for use in intermediate and senior mathematics classrooms is provided in *Leading Math Success*, pp. 48–49.

8. *Leading Math Success*, p. 42

Collaborative learning enhances students' understanding of mathematics. Working cooperatively in groups reduces isolation and provides students with opportunities to share ideas and communicate their thinking in a supportive environment as they work together towards a common goal. Communication and the connections among ideas that emerge as students interact with one another enhance the quality of student learning.<sup>9</sup>

## PLANNING MATHEMATICS PROGRAMS FOR STUDENTS WITH SPECIAL EDUCATION NEEDS

Classroom teachers are the key educators of students who have special education needs. They have a responsibility to help all students learn, and they work collaboratively with special education teachers, where appropriate, to achieve this goal. *Special Education Transformation: The Report of the Co-Chairs with the Recommendations of the Working Table on Special Education, 2006* endorses a set of beliefs that should guide program planning for students with special education needs *in all disciplines*. Those beliefs are as follows:

- All students can succeed.
- Universal design and differentiated instruction are effective and interconnected means of meeting the learning or productivity needs of any group of students.
- Successful instructional practices are founded on evidence-based research, tempered by experience.
- Classroom teachers are key educators for a student's literacy and numeracy development.
- Each student has his or her own unique patterns of learning.
- Classroom teachers need the support of the larger community to create a learning environment that supports students with special education needs.
- Fairness is not sameness.

In any given classroom, students may demonstrate a wide range of learning styles and needs. Teachers plan programs that recognize this diversity and give students performance tasks that respect their particular abilities so that all students can derive the greatest possible benefit from the teaching and learning process. The use of flexible groupings for instruction and the provision of ongoing assessment are important elements of programs that accommodate a diversity of learning needs.

In planning mathematics courses for students with special education needs, teachers should begin by examining the current achievement level of the individual student, the strengths and learning needs of the student, and the knowledge and skills that all students are expected to demonstrate at the end of the course in order to determine which of the following options is appropriate for the student:

- no accommodations<sup>10</sup> or modifications; or
- accommodations only; or
- modified expectations, with the possibility of accommodations; or
- alternative expectations, which are not derived from the curriculum expectations for a course and which constitute alternative programs and/or courses.

9. *Leading Math Success*, p. 42

10. "Accommodations" refers to individualized teaching and assessment strategies, human supports, and/or individualized equipment.

If the student requires either accommodations or modified expectations, or both, the relevant information, as described in the following paragraphs, must be recorded in his or her Individual Education Plan (IEP). More detailed information about planning programs for students with special education needs, including students who require alternative programs and/or courses, can be found in *The Individual Education Plan (IEP): A Resource Guide, 2004* (referred to hereafter as the *IEP Resource Guide, 2004*). For a detailed discussion of the ministry's requirements for IEPs, see *Individual Education Plans: Standards for Development, Program Planning, and Implementation, 2000* (referred to hereafter as *IEP Standards, 2000*). (Both documents are available at <http://www.edu.gov.on.ca>.)

### Students Requiring Accommodations Only

Some students are able, with certain accommodations, to participate in the regular course curriculum and to demonstrate learning independently. Accommodations allow access to the course without any changes to the knowledge and skills the student is expected to demonstrate. The accommodations required to facilitate the student's learning must be identified in his or her IEP (see *IEP Standards, 2000*, page 11). A student's IEP is likely to reflect the same accommodations for many, or all, subjects or courses.

Providing accommodations to students with special education needs should be the first option considered in program planning. Instruction based on principles of universal design and differentiated instruction focuses on the provision of accommodations to meet the diverse needs of learners.

There are three types of accommodations:

- *Instructional accommodations* are changes in teaching strategies, including styles of presentation, methods of organization, or use of technology and multimedia.
- *Environmental accommodations* are changes that the student may require in the classroom and/or school environment, such as preferential seating or special lighting.
- *Assessment accommodations* are changes in assessment procedures that enable the student to demonstrate his or her learning, such as allowing additional time to complete tests or assignments or permitting oral responses to test questions (see page 29 of the *IEP Resource Guide, 2004*, for more examples).

If a student requires "accommodations only" in mathematics courses, assessment and evaluation of his or her achievement will be based on the appropriate course curriculum expectations and the achievement levels outlined in this document. The IEP box on the student's Provincial Report Card will not be checked, and no information on the provision of accommodations will be included.

### Students Requiring Modified Expectations

Some students will require modified expectations, which differ from the regular course expectations. For most students, modified expectations will be based on the regular course curriculum, with changes in the number and/or complexity of the expectations. Modified expectations represent specific, realistic, observable, and measurable achievements and describe specific knowledge and/or skills that the student can demonstrate independently, given the appropriate assessment accommodations.

It is important to monitor, and to reflect clearly in the student's IEP, the extent to which expectations have been modified. As noted in Section 7.12 of the ministry's policy document *Ontario Secondary Schools, Grades 9 to 12: Program and Diploma Requirements, 1999*, the principal will determine whether achievement of the modified expectations constitutes successful completion of the course, and will decide whether the student is eligible to receive a credit for the course. This decision must be communicated to the parents and the student.

When a student is expected to achieve most of the curriculum expectations for the course, the modified expectations should identify *how the required knowledge and skills differ from those identified in the course expectations*. When modifications are so extensive that achievement of the learning expectations (knowledge, skills, and performance tasks) is not likely to result in a credit, the expectations should *specify the precise requirements or tasks on which the student's performance will be evaluated* and which will be used to generate the course mark recorded on the Provincial Report Card.

Modified expectations indicate the knowledge and/or skills the student is expected to demonstrate and have assessed *in each reporting period (IEP Standards, 2000, pages 10 and 11)*. The student's learning expectations must be reviewed in relation to the student's progress at least once every reporting period, and must be updated as necessary (*IEP Standards, 2000, page 11*).

If a student requires modified expectations in mathematics courses, assessment and evaluation of his or her achievement will be based on the learning expectations identified in the IEP and on the achievement levels outlined in this document. If some of the student's learning expectations for a course are modified but the student is working towards a credit for the course, it is sufficient simply to check the IEP box on the Provincial Report Card. If, however, the student's learning expectations are modified to such an extent that the principal deems that a credit will not be granted for the course, the IEP box must be checked and the appropriate statement from the *Guide to the Provincial Report Card, Grades 9–12, 1999* (page 8) must be inserted. The teacher's comments should include relevant information on the student's demonstrated learning of the modified expectations, as well as next steps for the student's learning in the course.

## PROGRAM CONSIDERATIONS FOR ENGLISH LANGUAGE LEARNERS

Young people whose first language is not English enter Ontario secondary schools with diverse linguistic and cultural backgrounds. Some English language learners may have experience of highly sophisticated educational systems, while others may have come from regions where access to formal schooling was limited. All of these students bring a rich array of background knowledge and experience to the classroom, and all teachers must share in the responsibility for their English-language development.

Teachers of mathematics must incorporate appropriate adaptations and strategies for instruction and assessment to facilitate the success of the English language learners in their classrooms. These adaptations and strategies include:

- modification of some or all of the course expectations so that they are challenging but attainable for the learner at his or her present level of English proficiency, given the necessary support from the teacher;

- use of a variety of instructional strategies (e.g., extensive use of visual cues, scaffolding, manipulatives, pictures, diagrams, graphic organizers; attention to clarity of instructions);
- modelling of preferred ways of working in mathematics; previewing of textbooks; pre-teaching of key vocabulary; peer tutoring; strategic use of students' first languages);
- use of a variety of learning resources (e.g., visual material, simplified text, bilingual dictionaries, materials that reflect cultural diversity);
- use of assessment accommodations (e.g., granting of extra time; simplification of language used in problems and instructions; use of oral interviews, learning logs, portfolios, demonstrations, visual representations, and tasks requiring completion of graphic organizers or cloze sentences instead of tasks that depend heavily on proficiency in English).

When learning expectations in any course are modified for English language learners (whether or not the students are enrolled in an ESL or ELD course), this must be clearly indicated on the student's report card.

Although the degree of program adaptation required will decrease over time, students who are no longer receiving ESL or ELD support may still need some program adaptations to be successful.

For further information on supporting English language learners, refer to *The Ontario Curriculum, Grades 9 to 12: English As a Second Language and English Literacy Development, 2007* and the resource guide *Many Roots Many Voices: Supporting English Language Learners in Every Classroom* (Ministry of Education, 2005).

## ANTIDISCRIMINATION EDUCATION IN MATHEMATICS

To ensure that all students in the province have an equal opportunity to achieve their full potential, the curriculum must be free from bias, and all students must be provided with a safe and secure environment, characterized by respect for others, that allows them to participate fully and responsibly in the educational experience.

Learning activities and resources used to implement the curriculum should be inclusive in nature, reflecting the range of experiences of students with varying backgrounds, abilities, interests, and learning styles. They should enable students to become more sensitive to the diverse cultures and perceptions of others, including Aboriginal peoples. By discussing aspects of the history of mathematics, teachers can help make students aware of the various cultural groups that have contributed to the evolution of mathematics over the centuries. Finally, students need to recognize that ordinary people use mathematics in a variety of everyday contexts, both at work and in their daily lives.

Connecting mathematical ideas to real-world situations through learning activities can enhance students' appreciation of the role of mathematics in human affairs, in areas including health, science, and the environment. Students can be made aware of the use of mathematics in contexts such as sampling and surveying and the use of statistics to analyse trends. Recognizing the importance of mathematics in such areas helps motivate students to learn and also provides a foundation for informed, responsible citizenship.

Teachers should have high expectations for all students. To achieve their mathematical potential, however, different students may need different kinds of support. Some boys, for example, may need additional support in developing their literacy skills in order to complete mathematical tasks effectively. For some girls, additional encouragement to envision themselves in careers involving mathematics may be beneficial. For example, teachers might consider providing strong role models in the form of female guest speakers who are mathematicians or who use mathematics in their careers.

## LITERACY AND INQUIRY/RESEARCH SKILLS

Literacy skills can play an important role in student success in mathematics courses. Many of the activities and tasks students undertake in mathematics courses involve the use of written, oral, and visual communication skills. For example, students use language to record their observations, to explain their reasoning when solving problems, to describe their inquiries in both informal and formal contexts, and to justify their results in small-group conversations, oral presentations, and written reports. The language of mathematics includes special terminology. The study of mathematics consequently encourages students to use language with greater care and precision and enhances their ability to communicate effectively.

The Ministry of Education has facilitated the development of materials to support literacy instruction across the curriculum. Helpful advice for integrating literacy instruction in mathematics courses may be found in the following resource documents:

- *Think Literacy: Cross-Curricular Approaches, Grades 7–12, 2003*
- *Think Literacy: Cross-Curricular Approaches, Grades 7–12 – Mathematics: Subject-Specific Examples, Grades 10–12, 2005*

In all courses in mathematics, students will develop their ability to ask questions and to plan investigations to answer those questions and to solve related problems. Students need to learn a variety of research methods and inquiry approaches in order to carry out these investigations and to solve problems, and they need to be able to select the methods that are most appropriate for a particular inquiry. Students learn how to locate relevant information from a variety of sources, such as statistical databases, newspapers, and reports. As they advance through the grades, students will be expected to use such sources with increasing sophistication. They will also be expected to distinguish between primary and secondary sources, to determine their validity and relevance, and to use them in appropriate ways.

## **THE ROLE OF INFORMATION AND COMMUNICATION TECHNOLOGY IN MATHEMATICS**

Information and communication technologies (ICT) provide a range of tools that can significantly extend and enrich teachers' instructional strategies and support students' learning in mathematics. Teachers can use ICT tools and resources both for whole-class instruction and to design programs that meet diverse student needs. Technology can help to reduce the time spent on routine mathematical tasks, allowing students to devote more of their efforts to thinking and concept development. Useful ICT tools include simulations, multimedia resources, databases, sites that give access to large amounts of statistical data, and computer-assisted learning modules.

Applications such as databases, spreadsheets, dynamic geometry software, dynamic statistical software, graphing software, computer algebra systems (CAS), word-processing software, and presentation software can be used to support various methods of inquiry in mathematics. Technology also makes possible simulations of complex systems that can be useful for problem-solving purposes or when field studies on a particular topic are not feasible.

Information and communications technologies can be used in the classroom to connect students to other schools, at home and abroad, and to bring the global community into the local classroom.

Although the Internet is a powerful electronic learning tool, there are potential risks attached to its use. All students must be made aware of issues of Internet privacy, safety, and responsible use, as well as of the ways in which this technology is being abused – for example, when it is used to promote hatred.

Teachers, too, will find the various ICT tools useful in their teaching practice, both for whole class instruction and for the design of curriculum units that contain varied approaches to learning to meet diverse student needs.

## **CAREER EDUCATION IN MATHEMATICS**

Teachers can promote students' awareness of careers involving mathematics by exploring applications of concepts and providing opportunities for career-related project work. Such activities allow students the opportunity to investigate mathematics-related careers compatible with their interests, aspirations, and abilities.

Students should be made aware that mathematical literacy and problem solving are valuable assets in an ever-widening range of jobs and careers in today's society. The knowledge and skills students acquire in mathematics courses are useful in fields such as science, business, engineering, and computer studies; in the hospitality, recreation, and tourism industries; and in the technical trades.

## THE ONTARIO SKILLS PASSPORT AND ESSENTIAL SKILLS

Teachers planning programs in mathematics need to be aware of the purpose and benefits of the *Ontario Skills Passport* (OSP). The OSP is a bilingual web-based resource that enhances the relevancy of classroom learning for students and strengthens school-work connections. The OSP provides clear descriptions of Essential Skills such as Reading Text, Writing, Computer Use, Measurement and Calculation, and Problem Solving and includes an extensive database of occupation-specific workplace tasks that illustrate how workers use these skills on the job. The Essential Skills are transferable, in that they are used in virtually all occupations. The OSP also includes descriptions of important work habits, such as working safely, being reliable, and providing excellent customer service. The OSP is designed to help employers assess and record students' demonstration of these skills and work habits during their cooperative education placements. Students can use the OSP to identify the skills and work habits they already have, plan further skill development, and show employers what they can do.

The skills described in the OSP are the Essential Skills that the Government of Canada and other national and international agencies have identified and validated, through extensive research, as the skills needed for work, learning, and life. These Essential Skills provide the foundation for learning all other skills and enable people to evolve with their jobs and adapt to workplace change. For further information on the OSP and the Essential Skills, visit: <http://skills.edu.gov.on.ca>.

## COOPERATIVE EDUCATION AND OTHER FORMS OF EXPERIENTIAL LEARNING

Cooperative education and other workplace experiences, such as job shadowing, field trips, and work experience, enable students to apply the skills they have developed in the classroom to real-life activities. Cooperative education and other workplace experiences also help to broaden students' knowledge of employment opportunities in a wide range of fields, including science and technology, research in the social sciences and humanities, and many forms of business administration. In addition, students develop their understanding of workplace practices, certifications, and the nature of employer-employee relationships.

Cooperative education teachers can support students taking mathematics courses by maintaining links with community-based businesses and organizations, and with colleges and universities, to ensure students studying mathematics have access to hands-on experiences that will reinforce the knowledge and skills they have gained in school. Teachers of mathematics can support their students' learning by providing opportunities for experiential learning that will reinforce the knowledge and skills they have gained in school.

Health and safety issues must be addressed when learning involves cooperative education and other workplace experiences. Teachers who provide support for students in workplace learning placements need to assess placements for safety and ensure students understand the importance of issues relating to health and safety in the workplace. Before taking part in workplace learning experiences, students must acquire the knowledge and skills needed for safe participation. Students must understand their rights to privacy and confidentiality as outlined in the Freedom of Information and Protection of Privacy Act. They have the right to function in an environment free from abuse and harassment, and

they need to be aware of harassment and abuse issues in establishing boundaries for their own personal safety. They should be informed about school and community resources and school policies and reporting procedures with regard to all forms of abuse and harassment.

Policy/Program Memorandum No. 76A, “Workplace Safety and Insurance Coverage for Students in Work Education Programs” (September 2000), outlines procedures for ensuring the provision of Health and Safety Insurance Board coverage for students who are at least 14 years of age and are on placements of more than one day. (A one-day job-shadowing or job-twinning experience is treated as a field trip.) Teachers should also be aware of the minimum age requirements outlined in the Occupational Health and Safety Act for persons to be in or to be working in specific workplace settings.

All cooperative education and other workplace experiences will be provided in accordance with the ministry’s policy document entitled *Cooperative Education and Other Forms of Experiential Learning: Policies and Procedures for Ontario Secondary Schools, 2000*.

## **PLANNING PROGRAM PATHWAYS AND PROGRAMS LEADING TO A SPECIALIST HIGH-SKILLS MAJOR**

Mathematics courses are well suited for inclusion in programs leading to a Specialist High-Skills Major (SHSM) or in programs designed to provide pathways to particular apprenticeship or workplace destinations. In an SHSM program, mathematics courses can be bundled with other courses to provide the academic knowledge and skills important to particular industry sectors and required for success in the workplace and postsecondary education, including apprenticeship. Mathematics courses may also be combined with cooperative education credits to provide the workplace experience required for SHSM programs and for various program pathways to apprenticeship and workplace destinations. (SHSM programs would also include sector-specific learning opportunities offered by employers, skills-training centres, colleges, and community organizations.)

## **HEALTH AND SAFETY IN MATHEMATICS**

Although health and safety issues are not normally associated with mathematics, they may be important when learning involves fieldwork or investigations based on experimentation. Out-of-school fieldwork can provide an exciting and authentic dimension to students’ learning experiences. It also takes the teacher and students out of the predictable classroom environment and into unfamiliar settings. Teachers must preview and plan activities and expeditions carefully to protect students’ health and safety.



# COURSES



# Functions, Grade 11

University Preparation

MCR3U

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This course introduces the mathematical concept of the function by extending students' experiences with linear and quadratic relations. Students will investigate properties of discrete and continuous functions, including trigonometric and exponential functions; represent functions numerically, algebraically, and graphically; solve problems involving applications of functions; investigate inverse functions; and develop facility in determining equivalent algebraic expressions. Students will reason mathematically and communicate their thinking as they solve multi-step problems.

**Prerequisite:** Principles of Mathematics, Grade 10, Academic

## MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

- Problem Solving**
  - develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;
- Reasoning and Proving**
  - develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;
  - demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);
- Reflecting**
- Selecting Tools and Computational Strategies**
  - select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;
- Connecting**
  - make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);
- Representing**
  - create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;
- Communicating**
  - communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

# A. CHARACTERISTICS OF FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of functions, their representations, and their inverses, and make connections between the algebraic and graphical representations of functions using transformations;
2. determine the zeros and the maximum or minimum of a quadratic function, and solve problems involving quadratic functions, including problems arising from real-world applications;
3. demonstrate an understanding of equivalence as it relates to simplifying polynomial, radical, and rational expressions.

## SPECIFIC EXPECTATIONS

### 1. Representing Functions

By the end of this course, students will:

- 1.1 explain the meaning of the term *function*, and distinguish a function from a relation that is not a function, through investigation of linear and quadratic relations using a variety of representations (i.e., tables of values, mapping diagrams, graphs, function machines, equations) and strategies (e.g., identifying a one-to-one or many-to-one mapping; using the vertical-line test)  
  
*Sample problem:* Investigate, using numeric and graphical representations, whether the relation  $x = y^2$  is a function, and justify your reasoning.
- 1.2 represent linear and quadratic functions using function notation, given their equations, tables of values, or graphs, and substitute into and evaluate functions [e.g., evaluate  $f\left(\frac{1}{2}\right)$ , given  $f(x) = 2x^2 + 3x - 1$ ]
- 1.3 explain the meanings of the terms *domain* and *range*, through investigation using numeric, graphical, and algebraic representations of the functions  $f(x) = x$ ,  $f(x) = x^2$ ,  $f(x) = \sqrt{x}$ , and  $f(x) = \frac{1}{x}$ ; describe the domain and range of a function appropriately (e.g., for  $y = x^2 + 1$ , the domain is the set of real numbers, and the range is  $y \geq 1$ ); and explain any restrictions on the domain and range in contexts arising from real-world applications

*Sample problem:* A quadratic function represents the relationship between the height of a ball and the time elapsed since the ball was thrown. What physical factors will restrict the domain and range of the quadratic function?

- 1.4 relate the process of determining the inverse of a function to their understanding of reverse processes (e.g., applying inverse operations)
- 1.5 determine the numeric or graphical representation of the inverse of a linear or quadratic function, given the numeric, graphical, or algebraic representation of the function, and make connections, through investigation using a variety of tools (e.g., graphing technology, Mira, tracing paper), between the graph of a function and the graph of its inverse (e.g., the graph of the inverse is the reflection of the graph of the function in the line  $y = x$ )  
  
*Sample problem:* Given a graph and a table of values representing population over time, produce a table of values for the inverse and graph the inverse on a new set of axes.
- 1.6 determine, through investigation, the relationship between the domain and range of a function and the domain and range of the inverse relation, and determine whether or not the inverse relation is a function

*Sample problem:* Given the graph of  $f(x) = x^2$ , graph the inverse relation. Compare the domain and range of the function with the domain

and range of the inverse relation, and investigate connections to the domain and range of the functions  $g(x) = \sqrt{x}$  and  $h(x) = -\sqrt{x}$ .

- 1.7** determine, using function notation when appropriate, the algebraic representation of the inverse of a linear or quadratic function, given the algebraic representation of the function [e.g.,  $f(x) = (x - 2)^2 - 5$ ], and make connections, through investigation using a variety of tools (e.g., graphing technology, Mira, tracing paper), between the algebraic representations of a function and its inverse (e.g., the inverse of a linear function involves applying the inverse operations in the reverse order)

**Sample problem:** Given the equations of several linear functions, graph the functions and their inverses, determine the equations of the inverses, and look for patterns that connect the equation of each linear function with the equation of the inverse.

- 1.8** determine, through investigation using technology, the roles of the parameters  $a$ ,  $k$ ,  $d$ , and  $c$  in functions of the form  $y = af(k(x - d)) + c$ , and describe these roles in terms of transformations on the graphs of  $f(x) = x$ ,  $f(x) = x^2$ ,  $f(x) = \sqrt{x}$ , and  $f(x) = \frac{1}{x}$  (i.e., translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the  $x$ - and  $y$ -axes)

**Sample problem:** Investigate the graph  $f(x) = 3(x - d)^2 + 5$  for various values of  $d$ , using technology, and describe the effects of changing  $d$  in terms of a transformation.

- 1.9** sketch graphs of  $y = af(k(x - d)) + c$  by applying one or more transformations to the graphs of  $f(x) = x$ ,  $f(x) = x^2$ ,  $f(x) = \sqrt{x}$ , and  $f(x) = \frac{1}{x}$ , and state the domain and range of the transformed functions

**Sample problem:** Transform the graph of  $f(x)$  to sketch  $g(x)$ , and state the domain and range of each function, for the following:

$$f(x) = \sqrt{x}, g(x) = \sqrt{x - 4}; f(x) = \frac{1}{x},$$

$$g(x) = -\frac{1}{x + 1}.$$

## 2. Solving Problems Involving Quadratic Functions

By the end of this course, students will:

- 2.1** determine the number of zeros (i.e.,  $x$ -intercepts) of a quadratic function, using a variety of strategies (e.g., inspecting graphs; factoring; calculating the discriminant)

**Sample problem:** Investigate, using graphing technology and algebraic techniques, the transformations that affect the number of zeros for a given quadratic function.

- 2.2** determine the maximum or minimum value of a quadratic function whose equation is given in the form  $f(x) = ax^2 + bx + c$ , using an algebraic method (e.g., completing the square; factoring to determine the zeros and averaging the zeros)

**Sample problem:** Explain how partially factoring  $f(x) = 3x^2 - 6x + 5$  into the form  $f(x) = 3x(x - 2) + 5$  helps you determine the minimum of the function.

- 2.3** solve problems involving quadratic functions arising from real-world applications and represented using function notation

**Sample problem:** The profit,  $P(x)$ , of a video company, in thousands of dollars, is given by  $P(x) = -5x^2 + 550x - 5000$ , where  $x$  is the amount spent on advertising, in thousands of dollars. Determine the maximum profit that the company can make, and the amounts spent on advertising that will result in a profit and that will result in a profit of at least \$4 000 000.

- 2.4** determine, through investigation, the transformational relationship among the family of quadratic functions that have the same zeros, and determine the algebraic representation of a quadratic function, given the real roots of the corresponding quadratic equation and a point on the function

**Sample problem:** Determine the equation of the quadratic function that passes through  $(2, 5)$  if the roots of the corresponding quadratic equation are  $1 + \sqrt{5}$  and  $1 - \sqrt{5}$ .

**2.5** solve problems involving the intersection of a linear function and a quadratic function graphically and algebraically (e.g., determine the time when two identical cylindrical water tanks contain equal volumes of water, if one tank is being filled at a constant rate and the other is being emptied through a hole in the bottom)

*Sample problem:* Determine, through investigation, the equations of the lines that have a slope of 2 and that intersect the quadratic function  $f(x) = x(6 - x)$  once; twice; never.

### 3. Determining Equivalent Algebraic Expressions\*

By the end of this course, students will:

**3.1** simplify polynomial expressions by adding, subtracting, and multiplying

*Sample problem:* Write and simplify an expression for the volume of a cube with edge length  $2x + 1$ .

**3.2** verify, through investigation with and without technology, that  $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ ,  $a \geq 0$ ,  $b \geq 0$ , and use this relationship to simplify radicals (e.g.,  $\sqrt{24}$ ) and radical expressions obtained by adding, subtracting, and multiplying [e.g.,  $(2 + \sqrt{6})(3 - \sqrt{12})$ ]

**3.3** simplify rational expressions by adding, subtracting, multiplying, and dividing, and state the restrictions on the variable values

*Sample problem:* Simplify

$$\frac{2x}{4x^2 + 6x} - \frac{3}{2x + 3},$$

and state the restrictions on the variable.

**3.4** determine if two given algebraic expressions are equivalent (i.e., by simplifying; by substituting values)

*Sample problem:* Determine if the expressions

$$\frac{2x^2 - 4x - 6}{x + 1} \text{ and } 8x^2 - 2x(4x - 1) - 6$$

are equivalent.

\*The knowledge and skills described in the expectations in this section are to be introduced as needed, and applied and consolidated, as appropriate, in solving problems throughout the course.

## B. EXPONENTIAL FUNCTIONS

### OVERALL EXPECTATIONS

By the end of this course, students will:

1. evaluate powers with rational exponents, simplify expressions containing exponents, and describe properties of exponential functions represented in a variety of ways;
2. make connections between the numeric, graphical, and algebraic representations of exponential functions;
3. identify and represent exponential functions, and solve problems involving exponential functions, including problems arising from real-world applications.

### SPECIFIC EXPECTATIONS

#### 1. Representing Exponential Functions

By the end of this course, students will:

- 1.1 graph, with and without technology, an exponential relation, given its equation in the form  $y = a^x$  ( $a > 0$ ,  $a \neq 1$ ), define this relation as the function  $f(x) = a^x$ , and explain why it is a function
- 1.2 determine, through investigation using a variety of tools (e.g., calculator, paper and pencil, graphing technology) and strategies (e.g., patterning; finding values from a graph; interpreting the exponent laws), the value of a power with a rational exponent (i.e.,  $x^{\frac{m}{n}}$ , where  $x > 0$  and  $m$  and  $n$  are integers)

**Sample problem:** The exponent laws suggest that  $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^1$ . What value would you assign to  $4^{\frac{1}{2}}$ ? What value would you assign to  $27^{\frac{1}{3}}$ ? Explain your reasoning. Extend your reasoning to make a generalization about the meaning of  $x^{\frac{1}{n}}$ , where  $x > 0$  and  $n$  is a natural number.

- 1.3 simplify algebraic expressions containing integer and rational exponents [e.g.,  $(x^3) \div (x^{\frac{1}{2}})$ ,  $(x^6 y^3)^{\frac{1}{3}}$ ], and evaluate numeric expressions containing integer and rational exponents and rational bases [e.g.,  $2^{-3}$ ,  $(-6)^3$ ,  $4^{\frac{1}{2}}$ ,  $1.01^{120}$ ]

- 1.4 determine, through investigation, and describe key properties relating to domain and range, intercepts, increasing/decreasing intervals, and asymptotes (e.g., the domain is the set of real numbers; the range is the set of positive real numbers; the function either increases or decreases throughout its domain) for exponential functions represented in a variety of ways [e.g., tables of values, mapping diagrams, graphs, equations of the form  $f(x) = a^x$  ( $a > 0$ ,  $a \neq 1$ ), function machines]

**Sample problem:** Graph  $f(x) = 2^x$ ,  $g(x) = 3^x$ , and  $h(x) = 0.5^x$  on the same set of axes. Make comparisons between the graphs, and explain the relationship between the  $y$ -intercepts.

#### 2. Connecting Graphs and Equations of Exponential Functions

By the end of this course, students will:

- 2.1 distinguish exponential functions from linear and quadratic functions by making comparisons in a variety of ways (e.g., comparing rates of change using finite differences in tables of values; identifying a constant ratio in a table of values; inspecting graphs; comparing equations)

**Sample problem:** Explain in a variety of ways how you can distinguish the exponential function  $f(x) = 2^x$  from the quadratic function  $f(x) = x^2$  and the linear function  $f(x) = 2x$ .

**2.2** determine, through investigation using technology, the roles of the parameters  $a$ ,  $k$ ,  $d$ , and  $c$  in functions of the form  $y = af(k(x - d)) + c$ , and describe these roles in terms of transformations on the graph of  $f(x) = a^x$  ( $a > 0$ ,  $a \neq 1$ ) (i.e., translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the  $x$ - and  $y$ -axes)

**Sample problem:** Investigate the graph of  $f(x) = 3^{x-d} - 5$  for various values of  $d$ , using technology, and describe the effects of changing  $d$  in terms of a transformation.

**2.3** sketch graphs of  $y = af(k(x - d)) + c$  by applying one or more transformations to the graph of  $f(x) = a^x$  ( $a > 0$ ,  $a \neq 1$ ), and state the domain and range of the transformed functions

**Sample problem:** Transform the graph of  $f(x) = 3^x$  to sketch  $g(x) = 3^{-(x+1)} - 2$ , and state the domain and range of each function.

**2.4** determine, through investigation using technology, that the equation of a given exponential function can be expressed using different bases [e.g.,  $f(x) = 9^x$  can be expressed as  $f(x) = 3^{2x}$ ], and explain the connections between the equivalent forms in a variety of ways (e.g., comparing graphs; using transformations; using the exponent laws)

**2.5** represent an exponential function with an equation, given its graph or its properties

**Sample problem:** Write two equations to represent the same exponential function with a  $y$ -intercept of 5 and an asymptote at  $y = 3$ . Investigate whether other exponential functions have the same properties. Use transformations to explain your observations.

**Sample problem:** Collect data and graph the cooling curve representing the relationship between temperature and time for hot water cooling in a porcelain mug. Predict the shape of the cooling curve when hot water cools in an insulated mug. Test your prediction.

**3.2** identify exponential functions, including those that arise from real-world applications involving growth and decay (e.g., radioactive decay, population growth, cooling rates, pressure in a leaking tire), given various representations (i.e., tables of values, graphs, equations), and explain any restrictions that the context places on the domain and range (e.g., ambient temperature limits the range for a cooling curve)

**Sample problem:** Using data from Statistics Canada, investigate to determine if there was a period of time over which the increase in Canada's national debt could be modelled using an exponential function.

**3.3** solve problems using given graphs or equations of exponential functions arising from a variety of real-world applications (e.g., radioactive decay, population growth, height of a bouncing ball, compound interest) by interpreting the graphs or by substituting values for the exponent into the equations

**Sample problem:** The temperature of a cooling liquid over time can be modelled by the exponential function

$$T(x) = 60\left(\frac{1}{2}\right)^{\frac{x}{30}} + 20, \text{ where } T(x) \text{ is the}$$

temperature, in degrees Celsius, and  $x$  is the elapsed time, in minutes. Graph the function and determine how long it takes for the temperature to reach  $28^\circ\text{C}$ .

### 3. Solving Problems Involving Exponential Functions

By the end of this course, students will:

**3.1** collect data that can be modelled as an exponential function, through investigation with and without technology, from primary sources, using a variety of tools (e.g., concrete materials such as number cubes, coins; measurement tools such as electronic probes), or from secondary sources (e.g., websites such as Statistics Canada, E-STAT), and graph the data

# C. DISCRETE FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of recursive sequences, represent recursive sequences in a variety of ways, and make connections to Pascal's triangle;
2. demonstrate an understanding of the relationships involved in arithmetic and geometric sequences and series, and solve related problems;
3. make connections between sequences, series, and financial applications, and solve problems involving compound interest and ordinary annuities.

## SPECIFIC EXPECTATIONS

### 1. Representing Sequences

By the end of this course, students will:

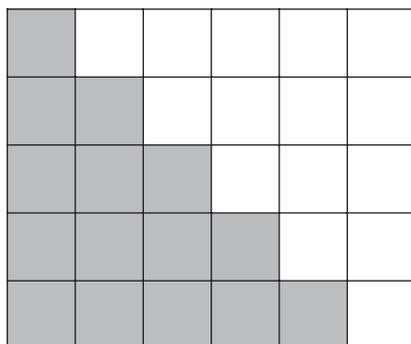
- 1.1 make connections between sequences and discrete functions, represent sequences using function notation, and distinguish between a discrete function and a continuous function [e.g.,  $f(x) = 2x$ , where the domain is the set of natural numbers, is a discrete linear function and its graph is a set of equally spaced points;  $f(x) = 2x$ , where the domain is the set of real numbers, is a continuous linear function and its graph is a straight line]
- 1.2 determine and describe (e.g., in words; using flow charts) a recursive procedure for generating a sequence, given the initial terms (e.g., 1, 3, 6, 10, 15, 21, ...), and represent sequences as discrete functions in a variety of ways (e.g., tables of values, graphs)
- 1.3 connect the formula for the  $n$ th term of a sequence to the representation in function notation, and write terms of a sequence given one of these representations or a recursion formula
- 1.4 represent a sequence algebraically using a recursion formula, function notation, or the formula for the  $n$ th term [e.g., represent 2, 4, 8, 16, 32, 64, ... as  $t_1 = 2$ ;  $t_n = 2t_{n-1}$ , as  $f(n) = 2^n$ , or as  $t_n = 2^n$ , or represent  $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \dots$  as  $t_1 = \frac{1}{2}$ ;  $t_n = t_{n-1} + \frac{1}{n(n+1)}$ , as  $f(n) = \frac{n}{n+1}$ , or as  $t_n = \frac{n}{n+1}$ , where  $n$  is a natural number], and describe the information that can be obtained by inspecting each representation (e.g., function notation or the formula for the  $n$ th term may show the type of function; a recursion formula shows the relationship between terms)  
*Sample problem:* Represent the sequence 0, 3, 8, 15, 24, 35, ... using a recursion formula, function notation, and the formula for the  $n$ th term. Explain why this sequence can be described as a discrete quadratic function. Explore how to identify a sequence as a discrete quadratic function by inspecting the recursion formula.
- 1.5 determine, through investigation, recursive patterns in the Fibonacci sequence, in related sequences, and in Pascal's triangle, and represent the patterns in a variety of ways (e.g., tables of values, algebraic notation)
- 1.6 determine, through investigation, and describe the relationship between Pascal's triangle and the expansion of binomials, and apply the relationship to expand binomials raised to whole-number exponents [e.g.,  $(1+x)^4$ ,  $(2x-1)^5$ ,  $(2x-y)^6$ ,  $(x^2+1)^5$ ]

## 2. Investigating Arithmetic and Geometric Sequences and Series

By the end of this course, students will:

- 2.1 identify sequences as arithmetic, geometric, or neither, given a numeric or algebraic representation
- 2.2 determine the formula for the general term of an arithmetic sequence [i.e.,  $t_n = a + (n - 1)d$ ] or geometric sequence (i.e.,  $t_n = ar^{n-1}$ ), through investigation using a variety of tools (e.g., linking cubes, algebra tiles, diagrams, calculators) and strategies (e.g., patterning; connecting the steps in a numerical example to the steps in the algebraic development), and apply the formula to calculate any term in a sequence
- 2.3 determine the formula for the sum of an arithmetic or geometric series, through investigation using a variety of tools (e.g., linking cubes, algebra tiles, diagrams, calculators) and strategies (e.g., patterning; connecting the steps in a numerical example to the steps in the algebraic development), and apply the formula to calculate the sum of a given number of consecutive terms

**Sample problem:** Given the following array built with grey and white connecting cubes, investigate how different ways of determining the total number of grey cubes can be used to evaluate the sum of the arithmetic series  $1 + 2 + 3 + 4 + 5$ . Extend the series, use patterning to make generalizations for finding the sum, and test the generalizations for other arithmetic series.



- 2.4 solve problems involving arithmetic and geometric sequences and series, including those arising from real-world applications

## 3. Solving Problems Involving Financial Applications

By the end of this course, students will:

- 3.1 make and describe connections between simple interest, arithmetic sequences, and linear growth, through investigation with technology (e.g., use a spreadsheet or graphing calculator to make simple interest calculations, determine first differences in the amounts over time, and graph amount versus time)
 

**Sample problem:** Describe an investment that could be represented by the function  $f(x) = 500(1 + 0.05x)$ .
- 3.2 make and describe connections between compound interest, geometric sequences, and exponential growth, through investigation with technology (e.g., use a spreadsheet to make compound interest calculations, determine finite differences in the amounts over time, and graph amount versus time)
 

**Sample problem:** Describe an investment that could be represented by the function  $f(x) = 500(1.05)^x$ .
- 3.3 solve problems, using a scientific calculator, that involve the calculation of the amount,  $A$  (also referred to as future value,  $FV$ ), the principal,  $P$  (also referred to as present value,  $PV$ ), or the interest rate per compounding period,  $i$ , using the compound interest formula in the form  $A = P(1 + i)^n$  [or  $FV = PV(1 + i)^n$ ]
 

**Sample problem:** Two investments are available, one at 6% compounded annually and the other at 6% compounded monthly. Investigate graphically the growth of each investment, and determine the interest earned from depositing \$1000 in each investment for 10 years.
- 3.4 determine, through investigation using technology (e.g., scientific calculator, the TVM Solver on a graphing calculator, online tools), the number of compounding periods,  $n$ , using the compound interest formula in the form  $A = P(1 + i)^n$  [or  $FV = PV(1 + i)^n$ ]; describe strategies (e.g., guessing and checking; using the power of a power rule for exponents; using graphs) for calculating this number; and solve related problems

- 3.5** explain the meaning of the term *annuity*, and determine the relationships between ordinary simple annuities (i.e., annuities in which payments are made at the *end* of each period, and compounding and payment periods are the same), geometric series, and exponential growth, through investigation with technology (e.g., use a spreadsheet to determine and graph the future value of an ordinary simple annuity for varying numbers of compounding periods; investigate how the contributions of each payment to the future value of an ordinary simple annuity are related to the terms of a geometric series)
- 3.6** determine, through investigation using technology (e.g., the TVM Solver on a graphing calculator, online tools), the effects of changing the conditions (i.e., the payments, the frequency of the payments, the interest rate, the compounding period) of ordinary simple annuities (e.g., long-term savings plans, loans)
- 3.7** solve problems, using technology (e.g., scientific calculator, spreadsheet, graphing calculator), that involve the amount, the present value, and the regular payment of an ordinary simple annuity (e.g., calculate the total interest paid over the life of a loan, using a spreadsheet, and compare the total interest with the original principal of the loan)
- Sample problem:** Compare the amounts at age 65 that would result from making an annual deposit of \$1000 starting at age 20, or from making an annual deposit of \$3000 starting at age 50, to an RRSP that earns 6% interest per annum, compounded annually. What is the total of the deposits in each situation?

# D. TRIGONOMETRIC FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. determine the values of the trigonometric ratios for angles less than  $360^\circ$ ; prove simple trigonometric identities; and solve problems using the primary trigonometric ratios, the sine law, and the cosine law;
2. demonstrate an understanding of periodic relationships and sinusoidal functions, and make connections between the numeric, graphical, and algebraic representations of sinusoidal functions;
3. identify and represent sinusoidal functions, and solve problems involving sinusoidal functions, including problems arising from real-world applications.

## SPECIFIC EXPECTATIONS

### 1. Determining and Applying Trigonometric Ratios

By the end of this course, students will:

- 1.1 determine the exact values of the sine, cosine, and tangent of the special angles:  $0^\circ$ ,  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , and  $90^\circ$
- 1.2 determine the values of the sine, cosine, and tangent of angles from  $0^\circ$  to  $360^\circ$ , through investigation using a variety of tools (e.g., dynamic geometry software, graphing tools) and strategies (e.g., applying the unit circle; examining angles related to special angles)
- 1.3 determine the measures of two angles from  $0^\circ$  to  $360^\circ$  for which the value of a given trigonometric ratio is the same
- 1.4 define the secant, cosecant, and cotangent ratios for angles in a right triangle in terms of the sides of the triangle (e.g.,  $\sec A = \frac{\text{hypotenuse}}{\text{adjacent}}$ ), and relate these ratios to the cosine, sine, and tangent ratios (e.g.,  $\sec A = \frac{1}{\cos A}$ )
- 1.5 prove simple trigonometric identities, using the Pythagorean identity  $\sin^2 x + \cos^2 x = 1$ ; the quotient identity  $\tan x = \frac{\sin x}{\cos x}$ ; and the reciprocal identities  $\sec x = \frac{1}{\cos x}$ ,  $\csc x = \frac{1}{\sin x}$ , and  $\cot x = \frac{1}{\tan x}$

*Sample problem:* Prove that

$$1 - \cos^2 x = \sin x \cos x \tan x.$$

- 1.6 pose problems involving right triangles and oblique triangles in two-dimensional settings, and solve these and other such problems using the primary trigonometric ratios, the cosine law, and the sine law (including the ambiguous case)
- 1.7 pose problems involving right triangles and oblique triangles in three-dimensional settings, and solve these and other such problems using the primary trigonometric ratios, the cosine law, and the sine law

*Sample problem:* Explain how a surveyor could find the height of a vertical cliff that is on the other side of a raging river, using a measuring tape, a theodolite, and some trigonometry. Determine what the surveyor might measure, and use hypothetical values for these data to calculate the height of the cliff.

### 2. Connecting Graphs and Equations of Sinusoidal Functions

By the end of this course, students will:

- 2.1 describe key properties (e.g., cycle, amplitude, period) of periodic functions arising from real-world applications (e.g., natural gas consumption in Ontario, tides in the Bay of Fundy), given a numeric or graphical representation

- 2.2** predict, by extrapolating, the future behaviour of a relationship modelled using a numeric or graphical representation of a periodic function (e.g., predicting hours of daylight on a particular date from previous measurements; predicting natural gas consumption in Ontario from previous consumption)
- 2.3** make connections between the sine ratio and the sine function and between the cosine ratio and the cosine function by graphing the relationship between angles from  $0^\circ$  to  $360^\circ$  and the corresponding sine ratios or cosine ratios, with or without technology (e.g., by generating a table of values using a calculator; by unwrapping the unit circle), defining this relationship as the function  $f(x) = \sin x$  or  $f(x) = \cos x$ , and explaining why the relationship is a function
- 2.4** sketch the graphs of  $f(x) = \sin x$  and  $f(x) = \cos x$  for angle measures expressed in degrees, and determine and describe their key properties (i.e., cycle, domain, range, intercepts, amplitude, period, maximum and minimum values, increasing/decreasing intervals)
- 2.5** determine, through investigation using technology, the roles of the parameters  $a$ ,  $k$ ,  $d$ , and  $c$  in functions of the form  $y = af(k(x - d)) + c$ , where  $f(x) = \sin x$  or  $f(x) = \cos x$  with angles expressed in degrees, and describe these roles in terms of transformations on the graphs of  $f(x) = \sin x$  and  $f(x) = \cos x$  (i.e., translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the  $x$ - and  $y$ -axes)

**Sample problem:** Investigate the graph  $f(x) = 2\sin(x - d) + 10$  for various values of  $d$ , using technology, and describe the effects of changing  $d$  in terms of a transformation.

- 2.6** determine the amplitude, period, phase shift, domain, and range of sinusoidal functions whose equations are given in the form  $f(x) = a\sin(k(x - d)) + c$  or  $f(x) = a\cos(k(x - d)) + c$
- 2.7** sketch graphs of  $y = af(k(x - d)) + c$  by applying one or more transformations to the graphs of  $f(x) = \sin x$  and  $f(x) = \cos x$ , and state the domain and range of the transformed functions

**Sample problem:** Transform the graph of  $f(x) = \cos x$  to sketch  $g(x) = 3\cos 2x - 1$ , and state the domain and range of each function.

- 2.8** represent a sinusoidal function with an equation, given its graph or its properties

**Sample problem:** A sinusoidal function has an amplitude of 2 units, a period of  $180^\circ$ , and a maximum at  $(0, 3)$ . Represent the function with an equation in two different ways.

### 3. Solving Problems Involving Sinusoidal Functions

By the end of this course, students will:

- 3.1** collect data that can be modelled as a sinusoidal function (e.g., voltage in an AC circuit, sound waves), through investigation with and without technology, from primary sources, using a variety of tools (e.g., concrete materials, measurement tools such as motion sensors), or from secondary sources (e.g., websites such as Statistics Canada, E-STAT), and graph the data

**Sample problem:** Measure and record distance–time data for a swinging pendulum, using a motion sensor or other measurement tools, and graph the data.

- 3.2** identify periodic and sinusoidal functions, including those that arise from real-world applications involving periodic phenomena, given various representations (i.e., tables of values, graphs, equations), and explain any restrictions that the context places on the domain and range

**Sample problem:** Using data from Statistics Canada, investigate to determine if there was a period of time over which changes in the population of Canadians aged 20–24 could be modelled using a sinusoidal function.

- 3.3** determine, through investigation, how sinusoidal functions can be used to model periodic phenomena that do not involve angles

**Sample problem:** Investigate, using graphing technology in degree mode, and explain how the function  $h(t) = 5\sin(30(t + 3))$  approximately models the relationship between the height and the time of day for a tide with an amplitude of 5 m, if high tide is at midnight.

- 3.4** predict the effects on a mathematical model (i.e., graph, equation) of an application involving periodic phenomena when the conditions in the application are varied (e.g., varying the conditions, such as speed and direction, when walking in a circle in front of a motion sensor)

**Sample problem:** The relationship between the height above the ground of a person riding a Ferris wheel and time can be modelled using a sinusoidal function. Describe the effect on this function if the platform from which the person enters the ride is raised by 1 m and if the Ferris wheel turns twice as fast.

- 3.5** pose problems based on applications involving a sinusoidal function, and solve these and other such problems by using a given graph or a graph generated with technology from a table of values or from its equation

**Sample problem:** The height above the ground of a rider on a Ferris wheel can be modelled by the sinusoidal function  $h(t) = 25 \sin(3(t - 30)) + 27$ , where  $h(t)$  is the height, in metres, and  $t$  is the time, in seconds. Graph the function, using graphing technology in degree mode, and determine the maximum and minimum heights of the rider, the height after 30 s, and the time required to complete one revolution.



# Functions and Applications, Grade 11

University/College Preparation

MCF3M

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This course introduces basic features of the function by extending students' experiences with quadratic relations. It focuses on quadratic, trigonometric, and exponential functions and their use in modelling real-world situations. Students will represent functions numerically, graphically, and algebraically; simplify expressions; solve equations; and solve problems relating to applications. Students will reason mathematically and communicate their thinking as they solve multi-step problems.

**Prerequisite:** Principles of Mathematics, Grade 10, Academic, or Foundations of Mathematics, Grade 10, Applied

## MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

- Problem Solving**
  - develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;
- Reasoning and Proving**
  - develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;
  - demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);
- Reflecting**
- Selecting Tools and Computational Strategies**
  - select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;
- Connecting**
  - make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);
- Representing**
  - create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;
- Communicating**
  - communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

# A. QUADRATIC FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. expand and simplify quadratic expressions, solve quadratic equations, and relate the roots of a quadratic equation to the corresponding graph;
2. demonstrate an understanding of functions, and make connections between the numeric, graphical, and algebraic representations of quadratic functions;
3. solve problems involving quadratic functions, including problems arising from real-world applications.

## SPECIFIC EXPECTATIONS

### 1. Solving Quadratic Equations

By the end of this course, students will:

- 1.1 pose problems involving quadratic relations arising from real-world applications and represented by tables of values and graphs, and solve these and other such problems (e.g., “From the graph of the height of a ball versus time, can you tell me how high the ball was thrown and the time when it hit the ground?”)
- 1.2 represent situations (e.g., the area of a picture frame of variable width) using quadratic expressions in one variable, and expand and simplify quadratic expressions in one variable [e.g.,  $2x(x + 4) - (x + 3)^2$ ]\*
- 1.3 factor quadratic expressions in one variable, including those for which  $a \neq 1$  (e.g.,  $3x^2 + 13x - 10$ ), differences of squares (e.g.,  $4x^2 - 25$ ), and perfect square trinomials (e.g.,  $9x^2 + 24x + 16$ ), by selecting and applying an appropriate strategy\*  
*Sample problem:* Factor  $2x^2 - 12x + 10$ .
- 1.4 solve quadratic equations by selecting and applying a factoring strategy
- 1.5 determine, through investigation, and describe the connection between the factors used in solving a quadratic equation and the  $x$ -intercepts of the graph of the corresponding quadratic relation

*Sample problem:* The profit,  $P$ , of a video company, in thousands of dollars, is given by  $P = -5x^2 + 550x - 5000$ , where  $x$  is the amount spent on advertising, in thousands of dollars. Determine, by factoring and by graphing, the amount spent on advertising that will result in a profit of \$0. Describe the connection between the two strategies.

- 1.6 explore the algebraic development of the quadratic formula (e.g., given the algebraic development, connect the steps to a numeric example; follow a demonstration of the algebraic development, with technology, such as computer algebra systems, or without technology [student reproduction of the development of the general case is not required]), and apply the formula to solve quadratic equations, using technology
- 1.7 relate the real roots of a quadratic equation to the  $x$ -intercepts of the corresponding graph, and connect the number of real roots to the value of the discriminant (e.g., there are no real roots and no  $x$ -intercepts if  $b^2 - 4ac < 0$ )
- 1.8 determine the real roots of a variety of quadratic equations (e.g.,  $100x^2 = 115x + 35$ ), and describe the advantages and disadvantages of each strategy (i.e., graphing; factoring; using the quadratic formula)  
*Sample problem:* Generate 10 quadratic equations by randomly selecting integer values for  $a$ ,  $b$ , and  $c$  in  $ax^2 + bx + c = 0$ . Solve the

\*The knowledge and skills described in this expectation may initially require the use of a variety of learning tools (e.g., computer algebra systems, algebra tiles, grid paper).

equations using the quadratic formula. How many of the equations could you solve by factoring?

## 2. Connecting Graphs and Equations of Quadratic Functions

By the end of this course, students will:

- 2.1** explain the meaning of the term *function*, and distinguish a function from a relation that is not a function, through investigation of linear and quadratic relations using a variety of representations (i.e., tables of values, mapping diagrams, graphs, function machines, equations) and strategies (e.g., using the vertical-line test)
- Sample problem:** Investigate, using numeric and graphical representations, whether the relation  $x = y^2$  is a function, and justify your reasoning.
- 2.2** substitute into and evaluate linear and quadratic functions represented using function notation [e.g., evaluate  $f\left(\frac{1}{2}\right)$ , given  $f(x) = 2x^2 + 3x - 1$ ], including functions arising from real-world applications
- Sample problem:** The relationship between the selling price of a sleeping bag,  $s$  dollars, and the revenue at that selling price,  $r(s)$  dollars, is represented by the function  $r(s) = -10s^2 + 1500s$ . Evaluate, interpret, and compare  $r(29.95)$ ,  $r(60.00)$ ,  $r(75.00)$ ,  $r(90.00)$ , and  $r(130.00)$ .
- 2.3** explain the meanings of the terms *domain* and *range*, through investigation using numeric, graphical, and algebraic representations of linear and quadratic functions, and describe the domain and range of a function appropriately (e.g., for  $y = x^2 + 1$ , the domain is the set of real numbers, and the range is  $y \geq 1$ )
- 2.4** explain any restrictions on the domain and the range of a quadratic function in contexts arising from real-world applications
- Sample problem:** A quadratic function represents the relationship between the height of a ball and the time elapsed since the ball was thrown. What physical factors will restrict the domain and range of the quadratic function?
- 2.5** determine, through investigation using technology, the roles of  $a$ ,  $h$ , and  $k$  in quadratic functions of the form  $f(x) = a(x - h)^2 + k$ , and describe these roles in terms of transformations on the graph of  $f(x) = x^2$  (i.e., translations; reflections in the  $x$ -axis; vertical stretches and compressions to and from the  $x$ -axis)
- Sample problem:** Investigate the graph  $f(x) = 3(x - h)^2 + 5$  for various values of  $h$ , using technology, and describe the effects of changing  $h$  in terms of a transformation.
- 2.6** sketch graphs of  $g(x) = a(x - h)^2 + k$  by applying one or more transformations to the graph of  $f(x) = x^2$
- Sample problem:** Transform the graph of  $f(x) = x^2$  to sketch the graphs of  $g(x) = x^2 - 4$  and  $h(x) = -2(x + 1)^2$ .
- 2.7** express the equation of a quadratic function in the standard form  $f(x) = ax^2 + bx + c$ , given the vertex form  $f(x) = a(x - h)^2 + k$ , and verify, using graphing technology, that these forms are equivalent representations
- Sample problem:** Given the vertex form  $f(x) = 3(x - 1)^2 + 4$ , express the equation in standard form. Use technology to compare the graphs of these two forms of the equation.
- 2.8** express the equation of a quadratic function in the vertex form  $f(x) = a(x - h)^2 + k$ , given the standard form  $f(x) = ax^2 + bx + c$ , by completing the square (e.g., using algebra tiles or diagrams; algebraically), including cases where  $\frac{b}{a}$  is a simple rational number (e.g.,  $\frac{1}{2}$ , 0.75), and verify, using graphing technology, that these forms are equivalent representations
- 2.9** sketch graphs of quadratic functions in the factored form  $f(x) = a(x - r)(x - s)$  by using the  $x$ -intercepts to determine the vertex
- 2.10** describe the information (e.g., maximum, intercepts) that can be obtained by inspecting the standard form  $f(x) = ax^2 + bx + c$ , the vertex form  $f(x) = a(x - h)^2 + k$ , and the factored form  $f(x) = a(x - r)(x - s)$  of a quadratic function
- 2.11** sketch the graph of a quadratic function whose equation is given in the standard form  $f(x) = ax^2 + bx + c$  by using a suitable strategy (e.g., completing the square and finding the vertex; factoring, if possible, to locate the  $x$ -intercepts), and identify the key features of the graph (e.g., the vertex, the  $x$ - and  $y$ -intercepts, the equation of the axis of symmetry, the intervals where the function is positive or negative, the intervals where the function is increasing or decreasing)

### 3. Solving Problems Involving Quadratic Functions

By the end of this course, students will:

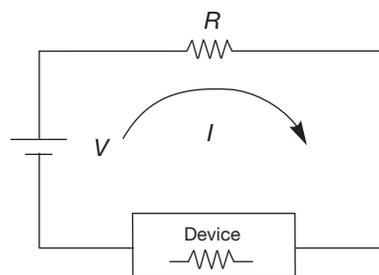
- 3.1** collect data that can be modelled as a quadratic function, through investigation with and without technology, from primary sources, using a variety of tools (e.g., concrete materials; measurement tools such as measuring tapes, electronic probes, motion sensors), or from secondary sources (e.g., websites such as Statistics Canada, E-STAT), and graph the data

*Sample problem:* When a  $3 \times 3 \times 3$  cube made up of  $1 \times 1 \times 1$  cubes is dipped into red paint, 6 of the smaller cubes will have 1 face painted. Investigate the number of smaller cubes with 1 face painted as a function of the edge length of the larger cube, and graph the function.

- 3.2** determine, through investigation using a variety of strategies (e.g., applying properties of quadratic functions such as the  $x$ -intercepts and the vertex; using transformations), the equation of the quadratic function that best models a suitable data set graphed on a scatter plot, and compare this equation to the equation of a curve of best fit generated with technology (e.g., graphing software, graphing calculator)

- 3.3** solve problems arising from real-world applications, given the algebraic representation of a quadratic function (e.g., given the equation of a quadratic function representing the height of a ball over elapsed time, answer questions that involve the maximum height of the ball, the length of time needed for the ball to touch the ground, and the time interval when the ball is higher than a given measurement)

*Sample problem:* In the following DC electrical circuit, the relationship between the power used by a device,  $P$  (in watts, W), the electric potential difference (voltage),  $V$  (in volts, V), the current,  $I$  (in amperes, A), and the resistance,  $R$  (in ohms,  $\Omega$ ), is represented by the formula  $P = IV - I^2R$ . Represent graphically and algebraically the relationship between the power and the current when the electric potential difference is 24 V and the resistance is  $1.5 \Omega$ . Determine the current needed in order for the device to use the maximum amount of power.



## B. EXPONENTIAL FUNCTIONS

### OVERALL EXPECTATIONS

By the end of this course, students will:

1. simplify and evaluate numerical expressions involving exponents, and make connections between the numeric, graphical, and algebraic representations of exponential functions;
2. identify and represent exponential functions, and solve problems involving exponential functions, including problems arising from real-world applications;
3. demonstrate an understanding of compound interest and annuities, and solve related problems.

### SPECIFIC EXPECTATIONS

#### 1. Connecting Graphs and Equations of Exponential Functions

By the end of this course, students will:

- 1.1** determine, through investigation using a variety of tools (e.g., calculator, paper and pencil, graphing technology) and strategies (e.g., patterning; finding values from a graph; interpreting the exponent laws), the value of a power with a rational exponent (i.e.,  $x^{\frac{m}{n}}$ , where  $x > 0$  and  $m$  and  $n$  are integers)

**Sample problem:** The exponent laws suggest that  $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^1$ . What value would you assign to  $4^{\frac{1}{2}}$ ? What value would you assign to  $27^{\frac{1}{3}}$ ? Explain your reasoning. Extend your reasoning to make a generalization about the meaning of  $x^{\frac{1}{n}}$ , where  $x > 0$  and  $n$  is a natural number.

- 1.2** evaluate, with and without technology, numerical expressions containing integer and rational exponents and rational bases [e.g.,  $2^{-3}$ ,  $(-6)^3$ ,  $4^{\frac{1}{2}}$ ,  $1.01^{120}$ ]
- 1.3** graph, with and without technology, an exponential relation, given its equation in the form  $y = a^x$  ( $a > 0$ ,  $a \neq 1$ ), define this relation as the function  $f(x) = a^x$ , and explain why it is a function
- 1.4** determine, through investigation, and describe key properties relating to domain and range, intercepts, increasing/decreasing intervals,

and asymptotes (e.g., the domain is the set of real numbers; the range is the set of positive real numbers; the function either increases or decreases throughout its domain) for exponential functions represented in a variety of ways [e.g., tables of values, mapping diagrams, graphs, equations of the form  $f(x) = a^x$  ( $a > 0$ ,  $a \neq 1$ ), function machines]

**Sample problem:** Graph  $f(x) = 2^x$ ,  $g(x) = 3^x$ , and  $h(x) = 0.5^x$  on the same set of axes. Make comparisons between the graphs, and explain the relationship between the  $y$ -intercepts.

- 1.5** determine, through investigation (e.g., by patterning with and without a calculator), the exponent rules for multiplying and dividing numeric expressions involving exponents [e.g.,  $(\frac{1}{2})^3 \times (\frac{1}{2})^2$ ], and the exponent rule for simplifying numerical expressions involving a power of a power [e.g.,  $(5^3)^2$ ], and use the rules to simplify numerical expressions containing integer exponents [e.g.,  $(2^3)(2^5) = 2^8$ ]
- 1.6** distinguish exponential functions from linear and quadratic functions by making comparisons in a variety of ways (e.g., comparing rates of change using finite differences in tables of values; identifying a constant ratio in a table of values; inspecting graphs; comparing equations), within the same context when possible (e.g., simple interest and compound interest, population growth)

**Sample problem:** Explain in a variety of ways how you can distinguish the exponential function  $f(x) = 2^x$  from the quadratic function  $f(x) = x^2$  and the linear function  $f(x) = 2x$ .

## 2. Solving Problems Involving Exponential Functions

By the end of this course, students will:

- 2.1** collect data that can be modelled as an exponential function, through investigation with and without technology, from primary sources, using a variety of tools (e.g., concrete materials such as number cubes, coins; measurement tools such as electronic probes), or from secondary sources (e.g., websites such as Statistics Canada, E-STAT), and graph the data

**Sample problem:** Collect data and graph the cooling curve representing the relationship between temperature and time for hot water cooling in a porcelain mug. Predict the shape of the cooling curve when hot water cools in an insulated mug. Test your prediction.

- 2.2** identify exponential functions, including those that arise from real-world applications involving growth and decay (e.g., radioactive decay, population growth, cooling rates, pressure in a leaking tire), given various representations (i.e., tables of values, graphs, equations), and explain any restrictions that the context places on the domain and range (e.g., ambient temperature limits the range for a cooling curve)

- 2.3** solve problems using given graphs or equations of exponential functions arising from a variety of real-world applications (e.g., radioactive decay, population growth, height of a bouncing ball, compound interest) by interpreting the graphs or by substituting values for the exponent into the equations

**Sample problem:** The temperature of a cooling liquid over time can be modelled by the exponential function  $T(x) = 60\left(\frac{1}{2}\right)^{\frac{x}{30}} + 20$ , where  $T(x)$  is the temperature, in degrees Celsius, and  $x$  is the elapsed time, in minutes. Graph the function and determine how long it takes for the temperature to reach 28°C.

## 3. Solving Financial Problems Involving Exponential Functions

By the end of this course, students will:

- 3.1** compare, using a table of values and graphs, the simple and compound interest earned for a given principal (i.e., investment) and a fixed interest rate over time

**Sample problem:** Compare, using tables of values and graphs, the amounts after each of the first five years for a \$1000 investment at 5% simple interest per annum and a \$1000 investment at 5% interest per annum, compounded annually.

- 3.2** solve problems, using a scientific calculator, that involve the calculation of the amount,  $A$  (also referred to as future value,  $FV$ ), and the principal,  $P$  (also referred to as present value,  $PV$ ), using the compound interest formula in the form  $A = P(1 + i)^n$  [or  $FV = PV(1 + i)^n$ ]

**Sample problem:** Calculate the amount if \$1000 is invested for three years at 6% per annum, compounded quarterly.

- 3.3** determine, through investigation (e.g., using spreadsheets and graphs), that compound interest is an example of exponential growth [e.g., the formulas for compound interest,  $A = P(1 + i)^n$ , and present value,  $PV = A(1 + i)^{-n}$ , are exponential functions, where the number of compounding periods,  $n$ , varies]

**Sample problem:** Describe an investment that could be represented by the function  $f(x) = 500(1.01)^x$ .

- 3.4** solve problems, using a TVM Solver on a graphing calculator or on a website, that involve the calculation of the interest rate per compounding period,  $i$ , or the number of compounding periods,  $n$ , in the compound interest formula  $A = P(1 + i)^n$  [or  $FV = PV(1 + i)^n$ ]

**Sample problem:** Use the TVM Solver in a graphing calculator to determine the time it takes to double an investment in an account that pays interest of 4% per annum, compounded semi-annually.

- 3.5** explain the meaning of the term *annuity*, through investigation of numeric and graphical representations using technology

**3.6** determine, through investigation using technology (e.g., the TVM Solver on a graphing calculator, online tools), the effects of changing the conditions (i.e., the payments, the frequency of the payments, the interest rate, the compounding period) of ordinary simple annuities (i.e., annuities in which payments are made at the *end* of each period, and the compounding period and the payment period are the same) (e.g., long-term savings plans, loans)

**Sample problem:** Compare the amounts at age 65 that would result from making an annual deposit of \$1000 starting at age 20, or from making an annual deposit of \$3000 starting at age 50, to an RRSP that earns 6% interest per annum, compounded annually. What is the total of the deposits in each situation?

**3.7** solve problems, using technology (e.g., scientific calculator, spreadsheet, graphing calculator), that involve the amount, the present value, and the regular payment of an ordinary simple annuity (e.g., calculate the total interest paid over the life of a loan, using a spreadsheet, and compare the total interest with the original principal of the loan)

# C. TRIGONOMETRIC FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. solve problems involving trigonometry in acute triangles using the sine law and the cosine law, including problems arising from real-world applications;
2. demonstrate an understanding of periodic relationships and the sine function, and make connections between the numeric, graphical, and algebraic representations of sine functions;
3. identify and represent sine functions, and solve problems involving sine functions, including problems arising from real-world applications.

## SPECIFIC EXPECTATIONS

### 1. Applying the Sine Law and the Cosine Law in Acute Triangles

By the end of this course, students will:

- 1.1 solve problems, including those that arise from real-world applications (e.g., surveying, navigation), by determining the measures of the sides and angles of right triangles using the primary trigonometric ratios
- 1.2 solve problems involving two right triangles in two dimensions  
*Sample problem:* A helicopter hovers 500 m above a long straight road. Ahead of the helicopter on the road are two trucks. The angles of depression of the two trucks from the helicopter are  $60^\circ$  and  $20^\circ$ . How far apart are the two trucks?
- 1.3 verify, through investigation using technology (e.g., dynamic geometry software, spreadsheet), the sine law and the cosine law (e.g., compare, using dynamic geometry software, the ratios  $\frac{a}{\sin A}$ ,  $\frac{b}{\sin B}$ , and  $\frac{c}{\sin C}$  in triangle  $ABC$  while dragging one of the vertices)
- 1.4 describe conditions that guide when it is appropriate to use the sine law or the cosine law, and use these laws to calculate sides and angles in acute triangles

- 1.5 solve problems that require the use of the sine law or the cosine law in acute triangles, including problems arising from real-world applications (e.g., surveying, navigation, building construction)

### 2. Connecting Graphs and Equations of Sine Functions

By the end of this course, students will:

- 2.1 describe key properties (e.g., cycle, amplitude, period) of periodic functions arising from real-world applications (e.g., natural gas consumption in Ontario, tides in the Bay of Fundy), given a numeric or graphical representation
- 2.2 predict, by extrapolating, the future behaviour of a relationship modelled using a numeric or graphical representation of a periodic function (e.g., predicting hours of daylight on a particular date from previous measurements; predicting natural gas consumption in Ontario from previous consumption)
- 2.3 make connections between the sine ratio and the sine function by graphing the relationship between angles from  $0^\circ$  to  $360^\circ$  and the corresponding sine ratios, with or without technology (e.g., by generating a table of values using a calculator; by unwrapping the unit circle), defining this relationship as the function  $f(x) = \sin x$ , and explaining why the relationship is a function

- 2.4** sketch the graph of  $f(x) = \sin x$  for angle measures expressed in degrees, and determine and describe its key properties (i.e., cycle, domain, range, intercepts, amplitude, period, maximum and minimum values, increasing/decreasing intervals)
- 2.5** make connections, through investigation with technology, between changes in a real-world situation that can be modelled using a periodic function and transformations of the corresponding graph (e.g., investigate the connection between variables for a swimmer swimming lengths of a pool and transformations of the graph of distance from the starting point versus time)
- Sample problem:** Generate the graph of a periodic function by walking a circle of 2-m diameter in front of a motion sensor. Describe how the following changes in the motion change the graph: starting at a different point on the circle; starting a greater distance from the motion sensor; changing direction; increasing the radius of the circle.
- 2.6** determine, through investigation using technology, the roles of the parameters  $a$ ,  $c$ , and  $d$  in functions in the form  $f(x) = a \sin x$ ,  $f(x) = \sin x + c$ , and  $f(x) = \sin(x - d)$ , and describe these roles in terms of transformations on the graph of  $f(x) = \sin x$  with angles expressed in degrees (i.e., translations; reflections in the  $x$ -axis; vertical stretches and compressions to and from the  $x$ -axis)
- 2.7** sketch graphs of  $f(x) = a \sin x$ ,  $f(x) = \sin x + c$ , and  $f(x) = \sin(x - d)$  by applying transformations to the graph of  $f(x) = \sin x$ , and state the domain and range of the transformed functions

**Sample problem:** Transform the graph of  $f(x) = \sin x$  to sketch the graphs of  $g(x) = -2\sin x$  and  $h(x) = \sin(x - 180^\circ)$ , and state the domain and range of each function.

### 3. Solving Problems Involving Sine Functions

By the end of this course, students will:

- 3.1** collect data that can be modelled as a sine function (e.g., voltage in an AC circuit, sound waves), through investigation with and without technology, from primary sources, using a variety of tools (e.g., concrete materials, measurement tools such as motion sensors), or from secondary sources (e.g., websites such as Statistics Canada, E-STAT), and graph the data
- Sample problem:** Measure and record distance-time data for a swinging pendulum, using a motion sensor or other measurement tools, and graph the data.
- 3.2** identify periodic and sinusoidal functions, including those that arise from real-world applications involving periodic phenomena, given various representations (i.e., tables of values, graphs, equations), and explain any restrictions that the context places on the domain and range
- 3.3** pose problems based on applications involving a sine function, and solve these and other such problems by using a given graph or a graph generated with technology from a table of values or from its equation

**Sample problem:** The height above the ground of a rider on a Ferris wheel can be modelled by the sine function  $h(x) = 25 \sin(x - 90^\circ) + 27$ , where  $h(x)$  is the height, in metres, and  $x$  is the angle, in degrees, that the radius from the centre of the ferris wheel to the rider makes with the horizontal. Graph the function, using graphing technology in degree mode, and determine the maximum and minimum heights of the rider and the measures of the angle when the height of the rider is 40 m.

# Foundations for College Mathematics, Grade 11

College Preparation

MBF3C

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This course enables students to broaden their understanding of mathematics as a problem-solving tool in the real world. Students will extend their understanding of quadratic relations; investigate situations involving exponential growth; solve problems involving compound interest; solve financial problems connected with vehicle ownership; develop their ability to reason by collecting, analysing, and evaluating data involving one variable; connect probability and statistics; and solve problems in geometry and trigonometry. Students will consolidate their mathematical skills as they solve problems and communicate their thinking.

**Prerequisite:** Foundations of Mathematics, Grade 10, Applied

## MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

- Problem Solving**
  - develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;
- Reasoning and Proving**
  - develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;
  - demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);
- Reflecting**
- Selecting Tools and Computational Strategies**
  - select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;
- Connecting**
  - make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);
- Representing**
  - create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;
- Communicating**
  - communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

# A. MATHEMATICAL MODELS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. make connections between the numeric, graphical, and algebraic representations of quadratic relations, and use the connections to solve problems;
2. demonstrate an understanding of exponents, and make connections between the numeric, graphical, and algebraic representations of exponential relations;
3. describe and represent exponential relations, and solve problems involving exponential relations arising from real-world applications.

## SPECIFIC EXPECTATIONS

### 1. Connecting Graphs and Equations of Quadratic Relations

By the end of this course, students will:

- 1.1 construct tables of values and graph quadratic relations arising from real-world applications (e.g., dropping a ball from a given height; varying the edge length of a cube and observing the effect on the surface area of the cube)
- 1.2 determine and interpret meaningful values of the variables, given a graph of a quadratic relation arising from a real-world application  
*Sample problem:* Under certain conditions, there is a quadratic relation between the profit of a manufacturing company and the number of items it produces. Explain how you could interpret a graph of the relation to determine the numbers of items produced for which the company makes a profit and to determine the maximum profit the company can make.
- 1.3 determine, through investigation using technology, the roles of  $a$ ,  $h$ , and  $k$  in quadratic relations of the form  $y = a(x - h)^2 + k$ , and describe these roles in terms of transformations on the graph of  $y = x^2$  (i.e., translations; reflections in the  $x$ -axis; vertical stretches and compressions to and from the  $x$ -axis)  
*Sample problem:* Investigate the graph  $y = 3(x - h)^2 + 5$  for various values of  $h$ , using technology, and describe the effects of changing  $h$  in terms of a transformation.
- 1.4 sketch graphs of quadratic relations represented by the equation  $y = a(x - h)^2 + k$  (e.g., using the vertex and at least one point on each side of the vertex; applying one or more transformations to the graph of  $y = x^2$ )
- 1.5 expand and simplify quadratic expressions in one variable involving multiplying binomials [e.g.,  $(\frac{1}{2}x + 1)(3x - 2)$ ] or squaring a binomial [e.g.,  $5(3x - 1)^2$ ], using a variety of tools (e.g., paper and pencil, algebra tiles, computer algebra systems)
- 1.6 express the equation of a quadratic relation in the standard form  $y = ax^2 + bx + c$ , given the vertex form  $y = a(x - h)^2 + k$ , and verify, using graphing technology, that these forms are equivalent representations  
*Sample problem:* Given the vertex form  $y = 3(x - 1)^2 + 4$ , express the equation in standard form. Use technology to compare the graphs of these two forms of the equation.
- 1.7 factor trinomials of the form  $ax^2 + bx + c$ , where  $a = 1$  or where  $a$  is the common factor, by various methods
- 1.8 determine, through investigation, and describe the connection between the factors of a quadratic expression and the  $x$ -intercepts of the graph of the corresponding quadratic relation  
*Sample problem:* Investigate the relationship between the factored form of  $3x^2 + 15x + 12$  and the  $x$ -intercepts of  $y = 3x^2 + 15x + 12$ .

- 1.9 solve problems, using an appropriate strategy (i.e., factoring, graphing), given equations of quadratic relations, including those that arise from real-world applications (e.g., break-even point)

**Sample problem:** On planet X, the height,  $h$  metres, of an object fired upward from the ground at 48 m/s is described by the equation  $h = 48t - 16t^2$ , where  $t$  seconds is the time since the object was fired upward. Determine the maximum height of the object, the times at which the object is 32 m above the ground, and the time at which the object hits the ground.

## 2. Connecting Graphs and Equations of Exponential Relations

By the end of this course, students will:

- 2.1 determine, through investigation using a variety of tools and strategies (e.g., graphing with technology; looking for patterns in tables of values), and describe the meaning of negative exponents and of zero as an exponent
- 2.2 evaluate, with and without technology, numeric expressions containing integer exponents and rational bases (e.g.,  $2^{-3}$ ,  $6^3$ ,  $3456^0$ ,  $1.03^{10}$ )
- 2.3 determine, through investigation (e.g., by patterning with and without a calculator), the exponent rules for multiplying and dividing numerical expressions involving exponents [e.g.,  $(\frac{1}{2})^3 \times (\frac{1}{2})^2$ ], and the exponent rule for simplifying numerical expressions involving a power of a power [e.g.,  $(5^3)^2$ ]
- 2.4 graph simple exponential relations, using paper and pencil, given their equations [e.g.,  $y = 2^x$ ,  $y = 10^x$ ,  $y = (\frac{1}{2})^x$ ]
- 2.5 make and describe connections between representations of an exponential relation (i.e., numeric in a table of values; graphical; algebraic)
- 2.6 distinguish exponential relations from linear and quadratic relations by making comparisons in a variety of ways (e.g., comparing rates of change using finite differences in tables of values; inspecting graphs; comparing equations), within the same context when possible (e.g., simple interest and compound interest, population growth)

**Sample problem:** Explain in a variety of ways how you can distinguish exponential growth represented by  $y = 2^x$  from quadratic growth represented by  $y = x^2$  and linear growth represented by  $y = 2x$ .

## 3. Solving Problems Involving Exponential Relations

By the end of this course, students will:

- 3.1 collect data that can be modelled as an exponential relation, through investigation with and without technology, from primary sources, using a variety of tools (e.g., concrete materials such as number cubes, coins; measurement tools such as electronic probes), or from secondary sources (e.g., websites such as Statistics Canada, E-STAT), and graph the data

**Sample problem:** Collect data and graph the cooling curve representing the relationship between temperature and time for hot water cooling in a porcelain mug. Predict the shape of the cooling curve when hot water cools in an insulated mug. Test your prediction.

- 3.2 describe some characteristics of exponential relations arising from real-world applications (e.g., bacterial growth, drug absorption) by using tables of values (e.g., to show a constant ratio, or multiplicative growth or decay) and graphs (e.g., to show, with technology, that there is no maximum or minimum value)
- 3.3 pose problems involving exponential relations arising from a variety of real-world applications (e.g., population growth, radioactive decay, compound interest), and solve these and other such problems by using a given graph or a graph generated with technology from a given table of values or a given equation

**Sample problem:** Given a graph of the population of a bacterial colony versus time, determine the change in population in the first hour.

- 3.4 solve problems using given equations of exponential relations arising from a variety of real-world applications (e.g., radioactive decay, population growth, height of a bouncing ball, compound interest) by substituting values for the exponent into the equations

**Sample problem:** The height,  $h$  metres, of a ball after  $n$  bounces is given by the equation  $h = 2(0.6)^n$ . Determine the height of the ball after 3 bounces.

## B. PERSONAL FINANCE

### OVERALL EXPECTATIONS

By the end of this course, students will:

1. compare simple and compound interest, relate compound interest to exponential growth, and solve problems involving compound interest;
2. compare services available from financial institutions, and solve problems involving the cost of making purchases on credit;
3. interpret information about owning and operating a vehicle, and solve problems involving the associated costs.

### SPECIFIC EXPECTATIONS

#### 1. Solving Problems Involving Compound Interest

By the end of this course, students will:

- 1.1** determine, through investigation using technology, the compound interest for a given investment, using repeated calculations of simple interest, and compare, using a table of values and graphs, the simple and compound interest earned for a given principal (i.e., investment) and a fixed interest rate over time

*Sample problem:* Compare, using tables of values and graphs, the amounts after each of the first five years for a \$1000 investment at 5% simple interest per annum and a \$1000 investment at 5% interest per annum, compounded annually.

- 1.2** determine, through investigation (e.g., using spreadsheets and graphs), and describe the relationship between compound interest and exponential growth

- 1.3** solve problems, using a scientific calculator, that involve the calculation of the amount,  $A$  (also referred to as future value,  $FV$ ), and the principal,  $P$  (also referred to as present value,  $PV$ ), using the compound interest formula in the form  $A = P(1 + i)^n$  [or  $FV = PV(1 + i)^n$ ]

*Sample problem:* Calculate the amount if \$1000 is invested for 3 years at 6% per annum, compounded quarterly.

- 1.4** calculate the total interest earned on an investment or paid on a loan by determining the difference between the amount and the principal [e.g., using  $I = A - P$  (or  $I = FV - PV$ )]

- 1.5** solve problems, using a TVM Solver on a graphing calculator or on a website, that involve the calculation of the interest rate per compounding period,  $i$ , or the number of compounding periods,  $n$ , in the compound interest formula  $A = P(1 + i)^n$  [or  $FV = PV(1 + i)^n$ ]

*Sample problem:* Use the TVM Solver on a graphing calculator to determine the time it takes to double an investment in an account that pays interest of 4% per annum, compounded semi-annually.

- 1.6** determine, through investigation using technology (e.g., a TVM Solver on a graphing calculator or on a website), the effect on the future value of a compound interest investment or loan of changing the total length of time, the interest rate, or the compounding period

*Sample problem:* Investigate whether doubling the interest rate will halve the time it takes for an investment to double.

## 2. Comparing Financial Services

By the end of this course, students will:

- 2.1 gather, interpret, and compare information about the various savings alternatives commonly available from financial institutions (e.g., savings and chequing accounts, term investments), the related costs (e.g., cost of cheques, monthly statement fees, early withdrawal penalties), and possible ways of reducing the costs (e.g., maintaining a minimum balance in a savings account; paying a monthly flat fee for a package of services)
- 2.2 gather and interpret information about investment alternatives (e.g., stocks, mutual funds, real estate, GICs, savings accounts), and compare the alternatives by considering the risk and the rate of return
- 2.3 gather, interpret, and compare information about the costs (e.g., user fees, annual fees, service charges, interest charges on overdue balances) and incentives (e.g., loyalty rewards; philanthropic incentives, such as support for Olympic athletes or a Red Cross disaster relief fund) associated with various credit cards and debit cards
- 2.4 gather, interpret, and compare information about current credit card interest rates and regulations, and determine, through investigation using technology, the effects of delayed payments on a credit card balance
- 2.5 solve problems involving applications of the compound interest formula to determine the cost of making a purchase on credit

**Sample problem:** Using information gathered about the interest rates and regulations for two different credit cards, compare the costs of purchasing a \$1500 computer with each card if the full amount is paid 55 days later.

## 3. Owning and Operating a Vehicle

By the end of this course, students will:

- 3.1 gather and interpret information about the procedures and costs involved in insuring a vehicle (e.g., car, motorcycle, snowmobile) and the factors affecting insurance rates (e.g., gender, age, driving record, model of vehicle, use of vehicle), and compare the insurance costs for different categories of drivers and for different vehicles
- 3.2 gather, interpret, and compare information about the procedures and costs (e.g., monthly payments, insurance, depreciation, maintenance, miscellaneous expenses) involved in buying or leasing a new vehicle or buying a used vehicle
- 3.3 solve problems, using technology (e.g., calculator, spreadsheet), that involve the fixed costs (e.g., licence fee, insurance) and variable costs (e.g., maintenance, fuel) of owning and operating a vehicle

**Sample problem:** Use automobile insurance websites to investigate the degree to which the type of car and the age and gender of the driver affect insurance rates.

**Sample problem:** Compare the costs of buying a new car, leasing the same car, and buying an older model of the same car.

**Sample problem:** The rate at which a car consumes gasoline depends on the speed of the car. Use a given graph of gasoline consumption, in litres per 100 km, versus speed, in kilometres per hour, to determine how much gasoline is used to drive 500 km at speeds of 80 km/h, 100 km/h, and 120 km/h. Use the current price of gasoline to calculate the cost of driving 500 km at each of these speeds.

# C. GEOMETRY AND TRIGONOMETRY

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. represent, in a variety of ways, two-dimensional shapes and three-dimensional figures arising from real-world applications, and solve design problems;
2. solve problems involving trigonometry in acute triangles using the sine law and the cosine law, including problems arising from real-world applications.

## SPECIFIC EXPECTATIONS

### 1. Representing Two-Dimensional Shapes and Three-Dimensional Figures

By the end of this course, students will:

- 1.1 recognize and describe real-world applications of geometric shapes and figures, through investigation (e.g., by importing digital photos into dynamic geometry software), in a variety of contexts (e.g., product design, architecture, fashion), and explain these applications (e.g., one reason that sewer covers are round is to prevent them from falling into the sewer during removal and replacement)

*Sample problem:* Explain why rectangular prisms are often used for packaging.

- 1.2 represent three-dimensional objects, using concrete materials and design or drawing software, in a variety of ways (e.g., orthographic projections [i.e., front, side, and top views], perspective isometric drawings, scale models)
- 1.3 create nets, plans, and patterns from physical models arising from a variety of real-world applications (e.g., fashion design, interior decorating, building construction), by applying the metric and imperial systems and using design or drawing software
- 1.4 solve design problems that satisfy given constraints (e.g., design a rectangular berm that would contain all the oil that could leak from a cylindrical storage tank of a given height and radius), using physical models (e.g., built from popsicle sticks, cardboard, duct tape) or

drawings (e.g., made using design or drawing software), and state any assumptions made

*Sample problem:* Design and construct a model boat that can carry the most pennies, using one sheet of 8.5 in. X 11 in. card stock, no more than five popsicle sticks, and some adhesive tape or glue.

### 2. Applying the Sine Law and the Cosine Law in Acute Triangles

By the end of this course, students will:

- 2.1 solve problems, including those that arise from real-world applications (e.g., surveying, navigation), by determining the measures of the sides and angles of right triangles using the primary trigonometric ratios
- 2.2 verify, through investigation using technology (e.g., dynamic geometry software, spreadsheet), the sine law and the cosine law (e.g., compare, using dynamic geometry software, the ratios  $\frac{a}{\sin A}$ ,  $\frac{b}{\sin B}$ , and  $\frac{c}{\sin C}$  in triangle  $ABC$  while dragging one of the vertices);
- 2.3 describe conditions that guide when it is appropriate to use the sine law or the cosine law, and use these laws to calculate sides and angles in acute triangles
- 2.4 solve problems that arise from real-world applications involving metric and imperial measurements and that require the use of the sine law or the cosine law in acute triangles

# D. DATA MANAGEMENT

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. solve problems involving one-variable data by collecting, organizing, analysing, and evaluating data;
2. determine and represent probability, and identify and interpret its applications.

## SPECIFIC EXPECTATIONS

### 1. Working With One-Variable Data

By the end of this course, students will:

- 1.1 identify situations involving one-variable data (i.e., data about the frequency of a given occurrence), and design questionnaires (e.g., for a store to determine which CDs to stock, for a radio station to choose which music to play) or experiments (e.g., counting, taking measurements) for gathering one-variable data, giving consideration to ethics, privacy, the need for honest responses, and possible sources of bias
 

**Sample problem:** One lane of a three-lane highway is being restricted to vehicles with at least two passengers to reduce traffic congestion. Design an experiment to collect one-variable data to decide whether traffic congestion is actually reduced.
- 1.2 collect one-variable data from secondary sources (e.g., Internet databases), and organize and store the data using a variety of tools (e.g., spreadsheets, dynamic statistical software)
- 1.3 explain the distinction between the terms *population* and *sample*, describe the characteristics of a good sample, and explain why sampling is necessary (e.g., time, cost, or physical constraints)
 

**Sample problem:** Explain the terms *sample* and *population* by giving examples within your school and your community.
- 1.4 describe and compare sampling techniques (e.g., random, stratified, clustered, convenience, voluntary); collect one-variable data from primary sources, using appropriate sampling techniques in a variety of real-world situations; and organize and store the data
- 1.5 identify different types of one-variable data (i.e., categorical, discrete, continuous), and represent the data, with and without technology, in appropriate graphical forms (e.g., histograms, bar graphs, circle graphs, pictographs)
- 1.6 identify and describe properties associated with common distributions of data (e.g., normal, bimodal, skewed)
- 1.7 calculate, using formulas and/or technology (e.g., dynamic statistical software, spreadsheet, graphing calculator), and interpret measures of central tendency (i.e., mean, median, mode) and measures of spread (i.e., range, standard deviation)
- 1.8 explain the appropriate use of measures of central tendency (i.e., mean, median, mode) and measures of spread (i.e., range, standard deviation)
 

**Sample problem:** Explain whether the mean or the median of your course marks would be the more appropriate representation of your achievement. Describe the additional information that the standard deviation of your course marks would provide.
- 1.9 compare two or more sets of one-variable data, using measures of central tendency and measures of spread
 

**Sample problem:** Use measures of central tendency and measures of spread to compare data that show the lifetime of an economy light bulb with data that show the lifetime of a long-life light bulb.
- 1.10 solve problems by interpreting and analysing one-variable data collected from secondary sources

## 2. Applying Probability

By the end of this course, students will:

- 2.1 identify examples of the use of probability in the media and various ways in which probability is represented (e.g., as a fraction, as a percent, as a decimal in the range 0 to 1)
- 2.2 determine the theoretical probability of an event (i.e., the ratio of the number of favourable outcomes to the total number of possible outcomes, where all outcomes are equally likely), and represent the probability in a variety of ways (e.g., as a fraction, as a percent, as a decimal in the range 0 to 1)
- 2.3 perform a probability experiment (e.g., tossing a coin several times), represent the results using a frequency distribution, and use the distribution to determine the experimental probability of an event
- 2.4 compare, through investigation, the theoretical probability of an event with the experimental probability, and explain why they might differ

*Sample problem:* If you toss 10 coins repeatedly, explain why 5 heads are unlikely to result from every toss.

- 2.5 determine, through investigation using class-generated data and technology-based simulation models (e.g., using a random-number generator on a spreadsheet or on a graphing calculator), the tendency of experimental probability to approach theoretical probability as the number of trials in an experiment increases (e.g., “If I simulate tossing a coin 1000 times using technology, the experimental probability that I calculate for tossing tails is likely to be closer to the theoretical probability than if I simulate tossing the coin only 10 times”)

*Sample problem:* Calculate the theoretical probability of rolling a 2 on a number cube. Simulate rolling a number cube, and use the simulation to calculate the experimental probability of rolling a 2 over 10, 20, 30, ..., 200 trials. Graph the experimental probability versus the number of trials, and describe any trend.

- 2.6 interpret information involving the use of probability and statistics in the media, and make connections between probability and statistics (e.g., statistics can be used to generate probabilities)



# Mathematics for Work and Everyday Life, Grade 11

Workplace Preparation

MEL3E

This course enables students to broaden their understanding of mathematics as it is applied in the workplace and daily life. Students will solve problems associated with earning money, paying taxes, and making purchases; apply calculations of simple and compound interest in saving, investing, and borrowing; and calculate the costs of transportation and travel in a variety of situations. Students will consolidate their mathematical skills as they solve problems and communicate their thinking.

**Prerequisite:** Principles of Mathematics, Grade 9, Academic, or Foundations of Mathematics, Grade 9, Applied, or a ministry-approved locally developed Grade 10 mathematics course

## MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

- Problem Solving**
  - develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;
- Reasoning and Proving**
  - develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;
  - demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);
- Reflecting**
- Selecting Tools and Computational Strategies**
  - select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;
- Connecting**
  - make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);
- Representing**
  - create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;
- Communicating**
  - communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

# A. EARNING AND PURCHASING

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. interpret information about different types of remuneration, and solve problems and make decisions involving different remuneration methods;
2. demonstrate an understanding of payroll deductions and their impact on purchasing power;
3. demonstrate an understanding of the factors and methods involved in making and justifying informed purchasing decisions.

## SPECIFIC EXPECTATIONS

### 1. Earning

By the end of this course, students will:

- 1.1 gather, interpret, and compare information about the components of total earnings (e.g., salary, benefits, vacation pay, profit-sharing) in different occupations
- 1.2 gather, interpret, and describe information about different remuneration methods (e.g., hourly rate, overtime rate, job or project rate, commission, salary, gratuities) and remuneration schedules (e.g., weekly, biweekly, semi-monthly, monthly)
- 1.3 describe the effects of different remuneration methods and schedules on decisions related to personal spending habits (e.g., the timing of a major purchase, the scheduling of mortgage payments and other bill payments)
- 1.4 solve problems, using technology (e.g., calculator, spreadsheet), and make decisions involving different remuneration methods and schedules

**Sample problem:** Two sales positions are available in sportswear stores. One pays an hourly rate of \$11.25 for 40 h per week. The other pays a weekly salary of \$375 for the same number of hours, plus a commission of 5% of sales. Under what conditions would each position be the better choice?

### 2. Describing Purchasing Power

By the end of this course, students will:

- 2.1 gather, interpret, and describe information about government payroll deductions (i.e., CPP, EI, income tax) and other payroll deductions (e.g., contributions to pension plans other than CPP; union dues; charitable donations; benefit-plan contributions)
- 2.2 estimate and compare, using current secondary data (e.g., federal tax tables), the percent of total earnings deducted through government payroll deductions for various benchmarks (e.g., \$15 000, \$20 000, \$25 000)
 

**Sample problem:** Compare the percentage of total earnings deducted through government payroll deductions for total earnings of \$15 000 and \$45 000.
- 2.3 describe the relationship between gross pay, net pay, and payroll deductions (i.e., net pay is gross pay less government payroll deductions and any other payroll deductions), and estimate net pay in various situations
- 2.4 describe and compare the purchasing power and living standards associated with relevant occupations of interest

### 3. Purchasing

By the end of this course, students will:

- 3.1** identify and describe various incentives in making purchasing decisions (e.g., 20% off;  $\frac{1}{3}$  off; buy 3 get 1 free; loyalty rewards; coupons; 0% financing)
- 3.2** estimate the sale price before taxes when making a purchase (e.g., estimate 25% off of \$38.99 as 25% or  $\frac{1}{4}$  off of \$40, giving a discount of about \$10 and a sale price of approximately \$30; alternatively, estimate the same sale price as about  $\frac{3}{4}$  of \$40)
- 3.3** describe and compare a variety of strategies for estimating sales tax (e.g., estimate the sales tax on most purchases in Ontario by estimating 10% of the purchase price and adding about a third of this estimate, rather than estimating the PST and GST separately), and use a chosen strategy to estimate the after-tax cost of common items

*Sample problem:* You purchase three items for \$8.99 each and one item for \$4.99. Estimate the after-tax total.

- 3.4** calculate discounts, sale prices, and after-tax costs, using technology
- 3.5** identify forms of taxation built into the cost of an item or service (e.g., gasoline tax, tire tax)
- 3.6** estimate the change from an amount offered to pay a charge
- Sample problem:* Estimate the change from the \$20 offered to pay a charge of \$13.87.

- 3.7** make the correct change from an amount offered to pay a charge, using currency manipulatives

*Sample problem:* Use currency manipulatives to explain why someone might offer \$15.02, rather than \$15.00, to pay a charge of \$13.87.

- 3.8** compare the unit prices of related items to help determine the best buy
- Sample problem:* Investigate whether or not purchasing larger quantities always results in a lower unit price.
- 3.9** describe and compare, for different types of transactions, the extra costs that may be associated with making purchases (e.g., interest costs, exchange rates, shipping and handling costs, customs duty, insurance)

*Sample problem:* What are the various costs included in the final total for purchasing a digital audio player online from an American source? Using an online calculator, calculate the final cost, and describe how it compares with the cost of the purchase from a major retailer in Ontario.

- 3.10** make and justify a decision regarding the purchase of an item, using various criteria (e.g., extra costs, such as shipping costs and transaction fees; quality and quantity of the item; shelf life of the item; method of purchase, such as online versus local) under various circumstances (e.g., not having access to a vehicle; living in a remote community; having limited storage space)

*Sample problem:* I have to take 100 mL of a liquid vitamin supplement every morning. I can buy a 100 mL size for \$6.50 or a 500 mL size for \$25.00. If the supplement keeps in the refrigerator for only 72 h, investigate which size is the better buy. Explain your reasoning.

## B. SAVING, INVESTING, AND BORROWING

### OVERALL EXPECTATIONS

By the end of this course, students will:

1. describe and compare services available from financial institutions;
2. demonstrate an understanding of simple and compound interest, and solve problems involving related applications;
3. interpret information about different ways of borrowing and their associated costs, and make and justify informed borrowing decisions.

### SPECIFIC EXPECTATIONS

#### 1. Comparing Financial Services

By the end of this course, students will:

- 1.1 gather, interpret, and compare information about the various savings alternatives commonly available from financial institutions (e.g., savings and chequing accounts, term investments), the related costs (e.g., cost of cheques, monthly statement fees, early withdrawal penalties), and possible ways of reducing the costs (e.g., maintaining a minimum balance in a savings account; paying a monthly flat fee for a package of services)
- 1.2 gather, interpret, and compare information about the costs (e.g., user fees, annual fees, service charges, interest charges on overdue balances) and incentives (e.g., loyalty rewards; philanthropic incentives, such as support for Olympic athletes or a Red Cross disaster relief fund) associated with various credit cards and debit cards
- 1.3 read and interpret transaction codes and entries from various financial statements (e.g., bank statement, credit card statement, passbook, automated banking machine printout, online banking statement, account activity report), and explain ways of using the information to manage personal finances

*Sample problem:* Examine a credit card statement and a bank statement for one individual, and comment on the individual's financial situation.

#### 2. Saving and Investing

By the end of this course, students will:

- 2.1 determine, through investigation using technology (e.g., calculator, spreadsheet), the effect on simple interest of changes in the principal, interest rate, or time, and solve problems involving applications of simple interest
- 2.2 determine, through investigation using technology, the compound interest for a given investment, using repeated calculations of simple interest for no more than 6 compounding periods  
*Sample problem:* Someone deposits \$5000 at 4% interest per annum, compounded semi-annually. How much interest accumulates in 3 years?
- 2.3 describe the relationship between simple interest and compound interest in various ways (i.e., orally, in writing, using tables and graphs)

**2.4** determine, through investigation using technology (e.g., a TVM Solver on a graphing calculator or on a website), the effect on the future value of a compound interest investment of changing the total length of time, the interest rate, or the compounding period

*Sample problem:* Compare the results at age 40 of making a deposit of \$1000 at age 20 or a deposit of \$2000 at age 30, if both investments pay 6% interest per annum, compounded monthly.

**2.5** solve problems, using technology, that involve applications of compound interest to saving and investing

### 3. Borrowing

By the end of this course, students will:

**3.1** gather, interpret, and compare information about the effects of carrying an outstanding balance on a credit card at current interest rates

*Sample problem:* Describe ways of minimizing the cost of carrying an outstanding balance on a credit card.

**3.2** gather, interpret, and compare information describing the features (e.g., interest rates, flexibility) and conditions (e.g., eligibility, required collateral) of various personal loans (e.g., student loan, car loan, “no interest” deferred-payment loan, loan to consolidate debt, loan drawn on a line of credit, payday or bridging loan)

**3.3** calculate, using technology (e.g., calculator, spreadsheet), the total interest paid over the life of a personal loan, given the principal, the length of the loan, and the periodic payments, and use the calculations to justify the choice of a personal loan

**3.4** determine, using a variety of tools (e.g., spreadsheet template, online amortization tables), the effect of the length of time taken to repay a loan on the principal and interest components of a personal loan repayment

**3.5** compare, using a variety of tools (e.g., spreadsheet template, online amortization tables), the effects of various payment periods (e.g., monthly, biweekly) on the length of time taken to repay a loan and on the total interest paid

**3.6** gather and interpret information about credit ratings, and describe the factors used to determine credit ratings and the consequences of a good or bad rating

**3.7** make and justify a decision to borrow, using various criteria (e.g., income, cost of borrowing, availability of an item, need for an item) under various circumstances (e.g., having a large existing debt, wanting to pursue an education or training opportunity, needing transportation to a new job, wanting to set up a business)

# C. TRANSPORTATION AND TRAVEL

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. interpret information about owning and operating a vehicle, and solve problems involving the associated costs;
2. plan and justify a route for a trip by automobile, and solve problems involving the associated costs;
3. interpret information about different modes of transportation, and solve related problems.

## SPECIFIC EXPECTATIONS

### 1. Owning and Operating a Vehicle

By the end of this course, students will:

- 1.1 gather and interpret information about the procedures (e.g., in the graduated licensing system) and costs (e.g., driver training; licensing fees) involved in obtaining an Ontario driver's licence, and the privileges and restrictions associated with having a driver's licence
- 1.2 gather and describe information about the procedures involved in buying or leasing a new vehicle or buying a used vehicle
- 1.3 gather and interpret information about the procedures and costs involved in insuring a vehicle (e.g., car, motorcycle, snowmobile) and the factors affecting insurance rates (e.g., gender, age, driving record, model of vehicle, use of vehicle), and compare the insurance costs for different categories of drivers and for different vehicles  
*Sample problem:* Use automobile insurance websites to investigate the degree to which the type of car and the age and gender of the driver affect insurance rates.
- 1.4 gather and interpret information about the costs (e.g., monthly payments, insurance, depreciation, maintenance, miscellaneous expenses) of purchasing or leasing a new vehicle or purchasing a used vehicle, and describe the conditions that favour each alternative

*Sample problem:* Compare the costs of buying a new car, leasing the same car, and buying an older model of the same car.

- 1.5 describe ways of failing to operate a vehicle responsibly (e.g., lack of maintenance, careless driving) and possible financial and non-financial consequences (e.g., legal costs, fines, higher insurance rates, demerit points, loss of driving privileges)
- 1.6 identify and describe costs (e.g., gas consumption, depreciation, insurance, maintenance) and benefits (e.g., convenience, increased profit) of owning and operating a vehicle for business  
*Sample problem:* Your employer pays 35 cents/km for you to use your car for work. Discuss how you would determine whether or not this is fair compensation.
- 1.7 solve problems, using technology (e.g., calculator, spreadsheet), that involve the fixed costs (e.g., licence fee, insurance) and variable costs (e.g., maintenance, fuel) of owning and operating a vehicle  
*Sample problem:* The rate at which a car consumes gasoline depends on the speed of the car. Use a given graph of gasoline consumption, in litres per 100 km, versus speed, in kilometres per hour, to determine how much gasoline is used to drive 500 km at speeds of 80 km/h, 100 km/h, and 120 km/h. Use the current price of gasoline to calculate the cost of driving 500 km at each of these speeds.

## 2. Travelling by Automobile

By the end of this course, students will:

- 2.1** determine distances represented on maps (e.g., provincial road map, local street map, Web-based maps), using given scales
- Sample problem:* Compare the driving distances between two points on the same map by two different routes.
- 2.2** plan and justify, orally or in writing, a route for a trip by automobile on the basis of a variety of factors (e.g., distances involved, the purpose of the trip, the time of year, the time of day, probable road conditions, personal priorities)
- 2.3** report, orally or in writing, on the estimated costs (e.g., gasoline, accommodation, food, entertainment, tolls, car rental) involved in a trip by automobile, using information from available sources (e.g., automobile association travel books, travel guides, the Internet)
- 2.4** solve problems involving the cost of travelling by automobile for personal or business purposes
- Sample problem:* Determine and justify a cost-effective delivery route for ten deliveries to be made in a given area over two days.

## 3. Comparing Modes of Transportation

By the end of this course, students will:

- 3.1** gather, interpret, and describe information about the impact (e.g., monetary, health, environmental) of daily travel (e.g., to work and/or school), using available means (e.g., car, taxi, motorcycle, public transportation, bicycle, walking)

*Sample problem:* Discuss the impact if 100 students decided to walk the 3-km distance to school instead of taking a school bus.

- 3.2** gather, interpret, and compare information about the costs (e.g., insurance, extra charges based on distance travelled) and conditions (e.g., one-way or return, drop-off time and location, age of the driver, required type of driver's licence) involved in renting a car, truck, or trailer, and use the information to justify a choice of rental vehicle
- Sample problem:* You want to rent a trailer or a truck to help you move to a new apartment. Investigate the costs and describe the conditions that favour each option.
- 3.3** gather, interpret, and describe information regarding routes, schedules, and fares for travel by airplane, train, or bus
- 3.4** solve problems involving the comparison of information concerning transportation by airplane, train, bus, and automobile in terms of various factors (e.g., cost, time, convenience)
- Sample problem:* Investigate the cost of shipping a computer from Thunder Bay to Windsor by airplane, train, or bus. Describe the conditions that favour each alternative.

# Advanced Functions, Grade 12

University Preparation

MHF4U

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This course extends students' experience with functions. Students will investigate the properties of polynomial, rational, logarithmic, and trigonometric functions; develop techniques for combining functions; broaden their understanding of rates of change; and develop facility in applying these concepts and skills. Students will also refine their use of the mathematical processes necessary for success in senior mathematics. This course is intended both for students taking the Calculus and Vectors course as a prerequisite for a university program and for those wishing to consolidate their understanding of mathematics before proceeding to any one of a variety of university programs.

**Prerequisite:** Functions, Grade 11, University Preparation, or Mathematics for College Technology, Grade 12, College Preparation

## MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

### Problem Solving

- develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;

### Reasoning and Proving

- develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;

### Reflecting

- demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);

### Selecting Tools and Computational Strategies

- select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;

### Connecting

- make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);

### Representing

- create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;

### Communicating

- communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

# A. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of the relationship between exponential expressions and logarithmic expressions, evaluate logarithms, and apply the laws of logarithms to simplify numeric expressions;
2. identify and describe some key features of the graphs of logarithmic functions, make connections among the numeric, graphical, and algebraic representations of logarithmic functions, and solve related problems graphically;
3. solve exponential and simple logarithmic equations in one variable algebraically, including those in problems arising from real-world applications.

## SPECIFIC EXPECTATIONS

### 1. Evaluating Logarithmic Expressions

By the end of this course, students will:

- 1.1 recognize the logarithm of a number to a given base as the exponent to which the base must be raised to get the number, recognize the operation of finding the logarithm to be the inverse operation (i.e., the undoing or reversing) of exponentiation, and evaluate simple logarithmic expressions

**Sample problem:** Why is it not possible to determine  $\log_{10}(-3)$  or  $\log_2 0$ ? Explain your reasoning.

- 1.2 determine, with technology, the approximate logarithm of a number to any base, including base 10 (e.g., by reasoning that  $\log_3 29$  is between 3 and 4 and using systematic trial to determine that  $\log_3 29$  is approximately 3.07)
- 1.3 make connections between related logarithmic and exponential equations (e.g.,  $\log_5 125 = 3$  can also be expressed as  $5^3 = 125$ ), and solve simple exponential equations by rewriting them in logarithmic form (e.g., solving  $3^x = 10$  by rewriting the equation as  $\log_3 10 = x$ )
- 1.4 make connections between the laws of exponents and the laws of logarithms [e.g., use the statement  $10^{a+b} = 10^a 10^b$  to deduce that  $\log_{10} x + \log_{10} y = \log_{10}(xy)$ ], verify the laws of logarithms with or without technology (e.g., use patterning to verify the quotient law for

logarithms by evaluating expressions such as  $\log_{10} 1000 - \log_{10} 100$  and then rewriting the answer as a logarithmic term to the same base), and use the laws of logarithms to simplify and evaluate numerical expressions

### 2. Connecting Graphs and Equations of Logarithmic Functions

By the end of this course, students will:

- 2.1 determine, through investigation with technology (e.g., graphing calculator, spreadsheet) and without technology, key features (i.e., vertical and horizontal asymptotes, domain and range, intercepts, increasing/decreasing behaviour) of the graphs of logarithmic functions of the form  $f(x) = \log_b x$ , and make connections between the algebraic and graphical representations of these logarithmic functions

**Sample problem:** Compare the key features of the graphs of  $f(x) = \log_2 x$ ,  $g(x) = \log_4 x$ , and  $h(x) = \log_8 x$  using graphing technology.

- 2.2 recognize the relationship between an exponential function and the corresponding logarithmic function to be that of a function and its inverse, deduce that the graph of a logarithmic function is the reflection of the graph of the corresponding exponential function in the line  $y = x$ , and verify the deduction using technology

**Sample problem:** Give examples to show that the inverse of a function is not necessarily a function. Use the key features of the graphs of logarithmic and exponential functions to give reasons why the inverse of an exponential function is a function.

- 2.3** determine, through investigation using technology, the roles of the parameters  $d$  and  $c$  in functions of the form  $y = \log_{10}(x - d) + c$  and the roles of the parameters  $a$  and  $k$  in functions of the form  $y = a\log_{10}(kx)$ , and describe these roles in terms of transformations on the graph of  $f(x) = \log_{10}x$  (i.e., vertical and horizontal translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the  $x$ - and  $y$ -axes)

**Sample problem:** Investigate the graphs of  $f(x) = \log_{10}(x) + c$ ,  $f(x) = \log_{10}(x - d)$ ,  $f(x) = a\log_{10}x$ , and  $f(x) = \log_{10}(kx)$  for various values of  $c$ ,  $d$ ,  $a$ , and  $k$ , using technology, describe the effects of changing these parameters in terms of transformations, and make connections to the transformations of other functions such as polynomial functions, exponential functions, and trigonometric functions.

- 2.4** pose problems based on real-world applications of exponential and logarithmic functions (e.g., exponential growth and decay, the Richter scale, the pH scale, the decibel scale), and solve these and other such problems by using a given graph or a graph generated with technology from a table of values or from its equation

**Sample problem:** The pH or acidity of a solution is given by the equation  $\text{pH} = -\log C$ , where  $C$  is the concentration of  $[\text{H}^+]$  ions in multiples of  $M = 1$  mol/L. Use graphing software to graph this function. What is the change in pH if the solution is diluted from a concentration of  $0.1M$  to a concentration of  $0.01M$ ? From  $0.001M$  to  $0.0001M$ ? Describe the change in pH when the concentration of any acidic solution is reduced to  $\frac{1}{10}$  of its original concentration. Rearrange the given equation to determine concentration as a function of pH.

### 3. Solving Exponential and Logarithmic Equations

By the end of this course, students will:

- 3.1** recognize equivalent algebraic expressions involving logarithms and exponents, and simplify expressions of these types

**Sample problem:** Sketch the graphs of  $f(x) = \log_{10}(100x)$  and  $g(x) = 2 + \log_{10}x$ , compare the graphs, and explain your findings algebraically.

- 3.2** solve exponential equations in one variable by determining a common base (e.g., solve  $4^x = 8^{x+3}$  by expressing each side as a power of 2) and by using logarithms (e.g., solve  $4^x = 8^{x+3}$  by taking the logarithm base 2 of both sides), recognizing that logarithms base 10 are commonly used (e.g., solving  $3^x = 7$  by taking the logarithm base 10 of both sides)

**Sample problem:** Solve  $300(1.05)^n = 600$  and  $2^{x+2} - 2^x = 12$  either by finding a common base or by taking logarithms, and explain your choice of method in each case.

- 3.3** solve simple logarithmic equations in one variable algebraically [e.g.,  $\log_3(5x + 6) = 2$ ,  $\log_{10}(x + 1) = 1$ ]

- 3.4** solve problems involving exponential and logarithmic equations algebraically, including problems arising from real-world applications

**Sample problem:** The pH or acidity of a solution is given by the equation  $\text{pH} = -\log C$ , where  $C$  is the concentration of  $[\text{H}^+]$  ions in multiples of  $M = 1$  mol/L. You are given a solution of hydrochloric acid with a pH of 1.7 and asked to increase the pH of the solution by 1.4. Determine how much you must dilute the solution. Does your answer differ if you start with a pH of 2.2?

# B. TRIGONOMETRIC FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of the meaning and application of radian measure;
2. make connections between trigonometric ratios and the graphical and algebraic representations of the corresponding trigonometric functions and between trigonometric functions and their reciprocals, and use these connections to solve problems;
3. solve problems involving trigonometric equations and prove trigonometric identities.

## SPECIFIC EXPECTATIONS

### 1. Understanding and Applying Radian Measure

By the end of this course, students will:

- 1.1 recognize the radian as an alternative unit to the degree for angle measurement, define the radian measure of an angle as the length of the arc that subtends this angle at the centre of a unit circle, and develop and apply the relationship between radian and degree measure
- 1.2 represent radian measure in terms of  $\pi$  (e.g.,  $\frac{\pi}{3}$  radians,  $2\pi$  radians) and as a rational number (e.g., 1.05 radians, 6.28 radians)
- 1.3 determine, with technology, the primary trigonometric ratios (i.e., sine, cosine, tangent) and the reciprocal trigonometric ratios (i.e., cosecant, secant, cotangent) of angles expressed in radian measure
- 1.4 determine, without technology, the exact values of the primary trigonometric ratios and the reciprocal trigonometric ratios for the special angles  $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ , and their multiples less than or equal to  $2\pi$

### 2. Connecting Graphs and Equations of Trigonometric Functions

By the end of this course, students will:

- 2.1 sketch the graphs of  $f(x) = \sin x$  and  $f(x) = \cos x$  for angle measures expressed in radians, and determine and describe some key properties (e.g., period of  $2\pi$ , amplitude of 1) in terms of radians
- 2.2 make connections between the tangent ratio and the tangent function by using technology to graph the relationship between angles in radians and their tangent ratios and defining this relationship as the function  $f(x) = \tan x$ , and describe key properties of the tangent function
- 2.3 graph, with technology and using the primary trigonometric functions, the reciprocal trigonometric functions (i.e., cosecant, secant, cotangent) for angle measures expressed in radians, determine and describe key properties of the reciprocal functions (e.g., state the domain, range, and period, and identify and explain the occurrence of asymptotes), and recognize notations used to represent the reciprocal functions [e.g., the reciprocal of  $f(x) = \sin x$  can be represented using  $\csc x$ ,  $\frac{1}{f(x)}$ , or  $\frac{1}{\sin x}$ , but not using  $f^{-1}(x)$  or  $\sin^{-1} x$ , which represent the inverse function]

2.4 determine the amplitude, period, and phase shift of sinusoidal functions whose equations are given in the form  $f(x) = a \sin(k(x - d)) + c$  or  $f(x) = a \cos(k(x - d)) + c$ , with angles expressed in radians

2.5 sketch graphs of  $y = a \sin(k(x - d)) + c$  and  $y = a \cos(k(x - d)) + c$  by applying transformations to the graphs of  $f(x) = \sin x$  and  $f(x) = \cos x$  with angles expressed in radians, and state the period, amplitude, and phase shift of the transformed functions

**Sample problem:** Transform the graph of  $f(x) = \cos x$  to sketch  $g(x) = 3 \cos(2x) - 1$ , and state the period, amplitude, and phase shift of each function.

2.6 represent a sinusoidal function with an equation, given its graph or its properties, with angles expressed in radians

**Sample problem:** A sinusoidal function has an amplitude of 2 units, a period of  $\pi$ , and a maximum at  $(0, 3)$ . Represent the function with an equation in two different ways.

2.7 pose problems based on applications involving a trigonometric function with domain expressed in radians (e.g., seasonal changes in temperature, heights of tides, hours of daylight, displacements for oscillating springs), and solve these and other such problems by using a given graph or a graph generated with or without technology from a table of values or from its equation

**Sample problem:** The population size,  $P$ , of owls (predators) in a certain region can be modelled by the function

$$P(t) = 1000 + 100 \sin\left(\frac{\pi t}{12}\right),$$

where  $t$  represents the time in months. The population size,  $p$ , of mice (prey) in the same region is given by

$$p(t) = 20\,000 + 4000 \cos\left(\frac{\pi t}{12}\right).$$

Sketch the graphs of these functions, and pose and solve problems involving the relationships between the two populations over time.

### 3. Solving Trigonometric Equations

By the end of this course, students will:

3.1 recognize equivalent trigonometric expressions [e.g., by using the angles in a right triangle to recognize that  $\sin x$  and  $\cos\left(\frac{\pi}{2} - x\right)$  are equivalent; by using transformations to recognize that  $\cos\left(x + \frac{\pi}{2}\right)$  and  $-\sin x$  are equivalent], and verify equivalence using graphing technology

3.2 explore the algebraic development of the compound angle formulas (e.g., verify the formulas in numerical examples, using technology; follow a demonstration of the algebraic development [student reproduction of the development of the general case is not required]), and use the formulas to determine exact values of trigonometric ratios [e.g., determining the exact value of  $\sin\left(\frac{\pi}{12}\right)$  by first rewriting it in terms of special angles as  $\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$ ]

3.3 recognize that trigonometric identities are equations that are true for every value in the domain (i.e., a counter-example can be used to show that an equation is not an identity), prove trigonometric identities through the application of reasoning skills, using a variety of relationships (e.g.,  $\tan x = \frac{\sin x}{\cos x}$ ;

$\sin^2 x + \cos^2 x = 1$ ; the reciprocal identities; the compound angle formulas), and verify identities using technology

**Sample problem:** Use the compound angle formulas to prove the double angle formulas.

3.4 solve linear and quadratic trigonometric equations, with and without graphing technology, for the domain of real values from  $0$  to  $2\pi$ , and solve related problems

**Sample problem:** Solve the following trigonometric equations for  $0 \leq x \leq 2\pi$ , and verify by graphing with technology:  $2 \sin x + 1 = 0$ ;  $2 \sin^2 x + \sin x - 1 = 0$ ;  $\sin x = \cos 2x$ ;

$$\cos 2x = \frac{1}{2}.$$

# C. POLYNOMIAL AND RATIONAL FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. identify and describe some key features of polynomial functions, and make connections between the numeric, graphical, and algebraic representations of polynomial functions;
2. identify and describe some key features of the graphs of rational functions, and represent rational functions graphically;
3. solve problems involving polynomial and simple rational equations graphically and algebraically;
4. demonstrate an understanding of solving polynomial and simple rational inequalities.

## SPECIFIC EXPECTATIONS

### 1. Connecting Graphs and Equations of Polynomial Functions

By the end of this course, students will:

- 1.1 recognize a polynomial expression (i.e., a series of terms where each term is the product of a constant and a power of  $x$  with a non-negative integral exponent, such as  $x^3 - 5x^2 + 2x - 1$ ); recognize the equation of a polynomial function, give reasons why it is a function, and identify linear and quadratic functions as examples of polynomial functions
- 1.2 compare, through investigation using graphing technology, the numeric, graphical, and algebraic representations of polynomial (i.e., linear, quadratic, cubic, quartic) functions (e.g., compare finite differences in tables of values; investigate the effect of the degree of a polynomial function on the shape of its graph and the maximum number of  $x$ -intercepts; investigate the effect of varying the sign of the leading coefficient on the end behaviour of the function for very large positive or negative  $x$ -values)
 

*Sample problem:* Investigate the maximum number of  $x$ -intercepts for linear, quadratic, cubic, and quartic functions using graphing technology.
- 1.3 describe key features of the graphs of polynomial functions (e.g., the domain and range, the shape of the graphs, the end behaviour of the functions for very large positive or negative  $x$ -values)
 

*Sample problem:* Describe and compare the key features of the graphs of the functions  $f(x) = x$ ,  $f(x) = x^2$ ,  $f(x) = x^3$ ,  $f(x) = x^3 + x^2$ , and  $f(x) = x^3 + x$ .
- 1.4 distinguish polynomial functions from sinusoidal and exponential functions [e.g.,  $f(x) = \sin x$ ,  $g(x) = 2^x$ ], and compare and contrast the graphs of various polynomial functions with the graphs of other types of functions
- 1.5 make connections, through investigation using graphing technology (e.g., dynamic geometry software), between a polynomial function given in factored form [e.g.,  $f(x) = 2(x - 3)(x + 2)(x - 1)$ ] and the  $x$ -intercepts of its graph, and sketch the graph of a polynomial function given in factored form using its key features (e.g., by determining intercepts and end behaviour; by locating positive and negative regions using test values between and on either side of the  $x$ -intercepts)
 

*Sample problem:* Investigate, using graphing technology, the  $x$ -intercepts and the shapes of the graphs of polynomial functions with

one or more repeated factors, for example,  $f(x) = (x - 2)(x - 3)$ ,  $f(x) = (x - 2)(x - 2)(x - 3)$ ,  $f(x) = (x - 2)(x - 2)(x - 2)(x - 3)$ , and  $f(x) = (x + 2)(x + 2)(x - 2)(x - 2)(x - 3)$ , by considering whether the factor is repeated an even or an odd number of times. Use your conclusions to sketch  $f(x) = (x + 1)(x + 1)(x - 3)(x - 3)$ , and verify using technology.

- 1.6** determine, through investigation using technology, the roles of the parameters  $a$ ,  $k$ ,  $d$ , and  $c$  in functions of the form  $y = af(k(x - d)) + c$ , and describe these roles in terms of transformations on the graphs of  $f(x) = x^3$  and  $f(x) = x^4$  (i.e., vertical and horizontal translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the  $x$ - and  $y$ -axes)

**Sample problem:** Investigate, using technology, the graph of  $f(x) = 2(x - d)^3 + c$  for various values of  $d$  and  $c$ , and describe the effects of changing  $d$  and  $c$  in terms of transformations.

- 1.7** determine an equation of a polynomial function that satisfies a given set of conditions (e.g., degree of the polynomial, intercepts, points on the function), using methods appropriate to the situation (e.g., using the  $x$ -intercepts of the function; using a trial-and-error process with a graphing calculator or graphing software; using finite differences), and recognize that there may be more than one polynomial function that can satisfy a given set of conditions (e.g., an infinite number of polynomial functions satisfy the condition that they have three given  $x$ -intercepts)

**Sample problem:** Determine an equation for a fifth-degree polynomial function that intersects the  $x$ -axis at only 5, 1, and  $-5$ , and sketch the graph of the function.

- 1.8** determine the equation of the family of polynomial functions with a given set of zeros and of the member of the family that passes through another given point [e.g., a family of polynomial functions of degree 3 with zeros 5,  $-3$ , and  $-2$  is defined by the equation  $f(x) = k(x - 5)(x + 3)(x + 2)$ , where  $k$  is a real number,  $k \neq 0$ ; the member of the family that passes through  $(-1, 24)$  is  $f(x) = -2(x - 5)(x + 3)(x + 2)$ ]

**Sample problem:** Investigate, using graphing technology, and determine a polynomial function that can be used to model the function  $f(x) = \sin x$  over the interval  $0 \leq x \leq 2\pi$ .

- 1.9** determine, through investigation, and compare the properties of even and odd polynomial functions [e.g., symmetry about the  $y$ -axis or the origin; the power of each term; the number of  $x$ -intercepts;  $f(x) = f(-x)$  or  $f(-x) = -f(x)$ ], and determine whether a given polynomial function is even, odd, or neither

**Sample problem:** Investigate numerically, graphically, and algebraically, with and without technology, the conditions under which an even function has an even number of  $x$ -intercepts.

## 2. Connecting Graphs and Equations of Rational Functions

By the end of this course, students will:

- 2.1** determine, through investigation with and without technology, key features (i.e., vertical and horizontal asymptotes, domain and range, intercepts, positive/negative intervals, increasing/decreasing intervals) of the graphs of rational functions that are the reciprocals of linear and quadratic functions, and make connections between the algebraic and graphical representations of these rational functions [e.g., make connections between  $f(x) = \frac{1}{x^2 - 4}$  and its graph by using graphing technology and by reasoning that there are vertical asymptotes at  $x = 2$  and  $x = -2$  and a horizontal asymptote at  $y = 0$  and that the function maintains the same sign as  $f(x) = x^2 - 4$ ]

**Sample problem:** Investigate, with technology, the key features of the graphs of families of rational functions of the form

$$f(x) = \frac{1}{x + n} \quad \text{and} \quad f(x) = \frac{1}{x^2 + n},$$

where  $n$  is an integer, and make connections between the equations and key features of the graphs.

- 2.2** determine, through investigation with and without technology, key features (i.e., vertical and horizontal asymptotes, domain and range, intercepts, positive/negative intervals, increasing/decreasing intervals) of the graphs of rational functions that have linear expressions in the numerator and denominator

$$\left[ \text{e.g., } f(x) = \frac{2x}{x - 3}, h(x) = \frac{x - 2}{3x + 4} \right], \text{ and}$$

make connections between the algebraic and graphical representations of these rational functions

**Sample problem:** Investigate, using graphing technology, key features of the graphs of the family of rational functions of the form

$f(x) = \frac{8x}{nx + 1}$  for  $n = 1, 2, 4,$  and  $8,$  and make connections between the equations and the asymptotes.

- 2.3** sketch the graph of a simple rational function using its key features, given the algebraic representation of the function

### 3. Solving Polynomial and Rational Equations

By the end of this course, students will:

- 3.1** make connections, through investigation using technology (e.g., computer algebra systems), between the polynomial function  $f(x)$ , the divisor  $x - a$ , the remainder from the division  $\frac{f(x)}{x - a}$ , and  $f(a)$  to verify the remainder theorem and the factor theorem

**Sample problem:** Divide

$f(x) = x^4 + 4x^3 - x^2 - 16x - 14$  by  $x - a$  for various integral values of  $a$  using a computer algebra system. Compare the remainder from each division with  $f(a)$ .

- 3.2** factor polynomial expressions in one variable, of degree no higher than four, by selecting and applying strategies (i.e., common factoring, difference of squares, trinomial factoring, factoring by grouping, remainder theorem, factor theorem)

**Sample problem:** Factor:  $x^3 + 2x^2 - x - 2$ ;  
 $x^4 - 6x^3 + 4x^2 + 6x - 5$ .

- 3.3** determine, through investigation using technology (e.g., graphing calculator, computer algebra systems), the connection between the real roots of a polynomial equation and the  $x$ -intercepts of the graph of the corresponding polynomial function, and describe this connection [e.g., the real roots of the equation  $x^4 - 13x^2 + 36 = 0$  are the  $x$ -intercepts of the graph of  $f(x) = x^4 - 13x^2 + 36$ ]

**Sample problem:** Describe the relationship between the  $x$ -intercepts of the graphs of linear and quadratic functions and the real roots of the corresponding equations. Investigate, using technology, whether this relationship exists for polynomial functions of higher degree.

- 3.4** solve polynomial equations in one variable, of degree no higher than four (e.g.,  $2x^3 - 3x^2 + 8x - 12 = 0$ ), by selecting and applying strategies (i.e., common factoring, difference of squares, trinomial factoring, factoring by grouping, remainder theorem, factor theorem), and verify solutions using technology (e.g., using computer algebra systems to determine the roots; using graphing technology to determine the  $x$ -intercepts of the graph of the corresponding polynomial function)

- 3.5** determine, through investigation using technology (e.g., graphing calculator, computer algebra systems), the connection between the real roots of a rational equation and the  $x$ -intercepts of the graph of the corresponding rational function, and describe this connection

[e.g., the real root of the equation  $\frac{x - 2}{x - 3} = 0$

is 2, which is the  $x$ -intercept of the function

$f(x) = \frac{x - 2}{x - 3}$ ; the equation  $\frac{1}{x - 3} = 0$  has no real roots, and the function  $f(x) = \frac{1}{x - 3}$  does not intersect the  $x$ -axis]

- 3.6** solve simple rational equations in one variable algebraically, and verify solutions using technology (e.g., using computer algebra systems to determine the roots; using graphing technology to determine the  $x$ -intercepts of the graph of the corresponding rational function)

- 3.7** solve problems involving applications of polynomial and simple rational functions and equations [e.g., problems involving the factor theorem or remainder theorem, such as determining the values of  $k$  for which the function  $f(x) = x^3 + 6x^2 + kx - 4$  gives the same remainder when divided by  $x - 1$  and  $x + 2$ ]

**Sample problem:** Use long division to express

the given function  $f(x) = \frac{x^2 + 3x - 5}{x - 1}$  as the

sum of a polynomial function and a rational function of the form  $\frac{A}{x - 1}$  (where  $A$  is a

constant), make a conjecture about the relationship between the given function and the polynomial function for very large positive and negative  $x$ -values, and verify your conjecture using graphing technology.

## 4. Solving Inequalities

By the end of this course, students will:

- 4.1** explain, for polynomial and simple rational functions, the difference between the solution to an equation in one variable and the solution to an inequality in one variable, and demonstrate that given solutions satisfy an inequality (e.g., demonstrate numerically and graphically that the solution to

$$\frac{1}{x+1} < 5 \text{ is } x < -1 \text{ or } x > -\frac{4}{5};$$

- 4.2** determine solutions to polynomial inequalities in one variable [e.g., solve  $f(x) \geq 0$ , where  $f(x) = x^3 - x^2 + 3x - 9$ ] and to simple rational inequalities in one variable by graphing the corresponding functions, using graphing technology, and identifying intervals for which  $x$  satisfies the inequalities

- 4.3** solve linear inequalities and factorable polynomial inequalities in one variable (e.g.,  $x^3 + x^2 > 0$ ) in a variety of ways (e.g., by determining intervals using  $x$ -intercepts and evaluating the corresponding function for a single  $x$ -value within each interval; by factoring the polynomial and identifying the conditions for which the product satisfies the inequality), and represent the solutions on a number line or algebraically (e.g., for the inequality  $x^4 - 5x^2 + 4 < 0$ , the solution represented algebraically is  $-2 < x < -1$  or  $1 < x < 2$ )

# D. CHARACTERISTICS OF FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of average and instantaneous rate of change, and determine, numerically and graphically, and interpret the average rate of change of a function over a given interval and the instantaneous rate of change of a function at a given point;
2. determine functions that result from the addition, subtraction, multiplication, and division of two functions and from the composition of two functions, describe some properties of the resulting functions, and solve related problems;
3. compare the characteristics of functions, and solve problems by modelling and reasoning with functions, including problems with solutions that are not accessible by standard algebraic techniques.

## SPECIFIC EXPECTATIONS

### 1. Understanding Rates of Change

By the end of this course, students will:

- 1.1 gather, interpret, and describe information about real-world applications of rates of change, and recognize different ways of representing rates of change (e.g., in words, numerically, graphically, algebraically)
- 1.2 recognize that the rate of change for a function is a comparison of changes in the dependent variable to changes in the independent variable, and distinguish situations in which the rate of change is zero, constant, or changing by examining applications, including those arising from real-world situations (e.g., rate of change of the area of a circle as the radius increases, inflation rates, the rising trend in graduation rates among Aboriginal youth, speed of a cruising aircraft, speed of a cyclist climbing a hill, infection rates)  
*Sample problem:* The population of bacteria in a sample is 250 000 at 1:00 p.m., 500 000 at 3:00 p.m., and 1 000 000 at 5:00 p.m. Compare methods used to calculate the change in the population and the rate of change in the population between 1:00 p.m. to 5:00 p.m. Is the rate of change constant? Explain your reasoning.
- 1.3 sketch a graph that represents a relationship involving rate of change, as described in words, and verify with technology (e.g., motion sensor) when possible

*Sample problem:* John rides his bicycle at a constant cruising speed along a flat road. He then decelerates (i.e., decreases speed) as he climbs a hill. At the top, he accelerates (i.e., increases speed) on a flat road back to his constant cruising speed, and he then accelerates down a hill. Finally, he comes to another hill and glides to a stop as he starts to climb. Sketch a graph of John's speed versus time and a graph of his distance travelled versus time.

- 1.4 calculate and interpret average rates of change of functions (e.g., linear, quadratic, exponential, sinusoidal) arising from real-world applications (e.g., in the natural, physical, and social sciences), given various representations of the functions (e.g., tables of values, graphs, equations)  
*Sample problem:* Fluorine-20 is a radioactive substance that decays over time. At time 0, the mass of a sample of the substance is 20 g. The mass decreases to 10 g after 11 s, to 5 g after 22 s, and to 2.5 g after 33 s. Compare the average rate of change over the 33-s interval with the average rate of change over consecutive 11-s intervals.
- 1.5 recognize examples of instantaneous rates of change arising from real-world situations, and make connections between instantaneous rates of change and average rates of change (e.g., an average rate of change can be used to approximate an instantaneous rate of change)

**Sample problem:** In general, does the speedometer of a car measure instantaneous rate of change (i.e., instantaneous speed) or average rate of change (i.e., average speed)? Describe situations in which the instantaneous speed and the average speed would be the same.

- 1.6 determine, through investigation using various representations of relationships (e.g., tables of values, graphs, equations), approximate instantaneous rates of change arising from real-world applications (e.g., in the natural, physical, and social sciences) by using average rates of change and reducing the interval over which the average rate of change is determined

**Sample problem:** The distance,  $d$  metres, travelled by a falling object in  $t$  seconds is represented by  $d = 5t^2$ . When  $t = 3$ , the instantaneous speed of the object is 30 m/s. Compare the average speeds over different time intervals starting at  $t = 3$  with the instantaneous speed when  $t = 3$ . Use your observations to select an interval that can be used to provide a good approximation of the instantaneous speed at  $t = 3$ .

- 1.7 make connections, through investigation, between the slope of a secant on the graph of a function (e.g., quadratic, exponential, sinusoidal) and the average rate of change of the function over an interval, and between the slope of the tangent to a point on the graph of a function and the instantaneous rate of change of the function at that point

**Sample problem:** Use tangents to investigate the behaviour of a function when the instantaneous rate of change is zero, positive, or negative.

- 1.8 determine, through investigation using a variety of tools and strategies (e.g., using a table of values to calculate slopes of secants or graphing secants and measuring their slopes with technology), the approximate slope of the tangent to a given point on the graph of a function (e.g., quadratic, exponential, sinusoidal) by using the slopes of secants through the given point (e.g., investigating the slopes of secants that approach the tangent at that point more and more closely), and make connections to average and instantaneous rates of change
- 1.9 solve problems involving average and instantaneous rates of change, including problems

arising from real-world applications, by using numerical and graphical methods (e.g., by using graphing technology to graph a tangent and measure its slope)

**Sample problem:** The height,  $h$  metres, of a ball above the ground can be modelled by the function  $h(t) = -5t^2 + 20t$ , where  $t$  is the time in seconds. Use average speeds to determine the approximate instantaneous speed at  $t = 3$ .

## 2. Combining Functions

By the end of this course, students will:

- 2.1 determine, through investigation using graphing technology, key features (e.g., domain, range, maximum/minimum points, number of zeros) of the graphs of functions created by adding, subtracting, multiplying, or dividing functions [e.g.,  $f(x) = 2^{-x} \sin 4x$ ,  $g(x) = x^2 + 2^x$ ,  $h(x) = \frac{\sin x}{\cos x}$ ], and describe factors that affect these properties

**Sample problem:** Investigate the effect of the behaviours of  $f(x) = \sin x$ ,  $f(x) = \sin 2x$ , and  $f(x) = \sin 4x$  on the shape of  $f(x) = \sin x + \sin 2x + \sin 4x$ .

- 2.2 recognize real-world applications of combinations of functions (e.g., the motion of a damped pendulum can be represented by a function that is the product of a trigonometric function and an exponential function; the frequencies of tones associated with the numbers on a telephone involve the addition of two trigonometric functions), and solve related problems graphically

**Sample problem:** The rate at which a contaminant leaves a storm sewer and enters a lake depends on two factors: the concentration of the contaminant in the water from the sewer and the rate at which the water leaves the sewer. Both of these factors vary with time. The concentration of the contaminant, in kilograms per cubic metre of water, is given by  $c(t) = t^2$ , where  $t$  is in seconds. The rate at which water leaves the sewer, in cubic metres per second, is given by  $w(t) = \frac{1}{t^4 + 10}$ . Determine the time at which the contaminant leaves the sewer and enters the lake at the maximum rate.

- 2.3** determine, through investigation, and explain some properties (i.e., odd, even, or neither; increasing/decreasing behaviours) of functions formed by adding, subtracting, multiplying, and dividing general functions [e.g.,  $f(x) + g(x)$ ,  $f(x)g(x)$ ]

**Sample problem:** Investigate algebraically, and verify numerically and graphically, whether the product of two functions is even or odd if the two functions are both even or both odd, or if one function is even and the other is odd.

- 2.4** determine the composition of two functions [i.e.,  $f(g(x))$ ] numerically (i.e., by using a table of values) and graphically, with technology, for functions represented in a variety of ways (e.g., function machines, graphs, equations), and interpret the composition of two functions in real-world applications

**Sample problem:** For a car travelling at a constant speed, the distance driven,  $d$  kilometres, is represented by  $d(t) = 80t$ , where  $t$  is the time in hours. The cost of gasoline, in dollars, for the drive is represented by  $C(d) = 0.09d$ . Determine numerically and interpret  $C(d(5))$ , and describe the relationship represented by  $C(d(t))$ .

- 2.5** determine algebraically the composition of two functions [i.e.,  $f(g(x))$ ], verify that  $f(g(x))$  is not always equal to  $g(f(x))$  [e.g., by determining  $f(g(x))$  and  $g(f(x))$ , given  $f(x) = x + 1$  and  $g(x) = 2x$ ], and state the domain [i.e., by defining  $f(g(x))$  for those  $x$ -values for which  $g(x)$  is defined and for which it is included in the domain of  $f(x)$ ] and the range of the composition of two functions

**Sample problem:** Determine  $f(g(x))$  and  $g(f(x))$  given  $f(x) = \cos x$  and  $g(x) = 2x + 1$ , state the domain and range of  $f(g(x))$  and  $g(f(x))$ , compare  $f(g(x))$  with  $g(f(x))$  algebraically, and verify numerically and graphically with technology.

- 2.6** solve problems involving the composition of two functions, including problems arising from real-world applications

**Sample problem:** The speed of a car,  $v$  kilometres per hour, at a time of  $t$  hours is represented by  $v(t) = 40 + 3t + t^2$ . The rate of gasoline consumption of the car,  $c$  litres per kilometre, at a speed of  $v$  kilometres per hour is represented by  $c(v) = \left(\frac{v}{500} - 0.1\right)^2 + 0.15$ .

Determine algebraically  $c(v(t))$ , the rate of gasoline consumption as a function of time. Determine, using technology, the time when the car is running most economically during a four-hour trip.

- 2.7** demonstrate, by giving examples for functions represented in a variety of ways (e.g., function machines, graphs, equations), the property that the composition of a function and its inverse function maps a number onto itself [i.e.,  $f^{-1}(f(x)) = x$  and  $f(f^{-1}(x)) = x$ ] demonstrate that the inverse function is the reverse process of the original function and that it undoes what the function does]

- 2.8** make connections, through investigation using technology, between transformations (i.e., vertical and horizontal translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the  $x$ - and  $y$ -axes) of simple functions  $f(x)$  [e.g.,  $f(x) = x^3 + 20$ ,  $f(x) = \sin x$ ,  $f(x) = \log x$ ] and the composition of these functions with a linear function of the form  $g(x) = A(x + B)$

**Sample problem:** Compare the graph of  $f(x) = x^2$  with the graphs of  $f(g(x))$  and  $g(f(x))$ , where  $g(x) = 2(x - d)$ , for various values of  $d$ . Describe the effects of  $d$  in terms of transformations of  $f(x)$ .

### 3. Using Function Models to Solve Problems

By the end of this course, students will:

- 3.1** compare, through investigation using a variety of tools and strategies (e.g., graphing with technology; comparing algebraic representations; comparing finite differences in tables of values) the characteristics (e.g., key features of the graphs, forms of the equations) of various functions (i.e., polynomial, rational, trigonometric, exponential, logarithmic)
- 3.2** solve graphically and numerically equations and inequalities whose solutions are not accessible by standard algebraic techniques
- Sample problem:** Solve:  $2x^2 < 2^x$ ;  $\cos x = x$ , with  $x$  in radians.
- 3.3** solve problems, using a variety of tools and strategies, including problems arising from real-world applications, by reasoning with functions and by applying concepts and procedures involving functions (e.g., by

constructing a function model from data, using the model to determine mathematical results, and interpreting and communicating the results within the context of the problem)

**Sample problem:** The pressure of a car tire with a slow leak is given in the following table of values:

Time, $t$ (min)	Pressure, $P$ (kPa)
0	400
5	335
10	295
15	255
20	225
25	195
30	170

Use technology to investigate linear, quadratic, and exponential models for the relationship of the tire pressure and time, and describe how well each model fits the data. Use each model to predict the pressure after 60 min. Which model gives the most realistic answer?

# Calculus and Vectors, Grade 12

University Preparation

MCV4U

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This course builds on students' previous experience with functions and their developing understanding of rates of change. Students will solve problems involving geometric and algebraic representations of vectors and representations of lines and planes in three-dimensional space; broaden their understanding of rates of change to include the derivatives of polynomial, sinusoidal, exponential, rational, and radical functions; and apply these concepts and skills to the modelling of real-world relationships. Students will also refine their use of the mathematical processes necessary for success in senior mathematics. This course is intended for students who choose to pursue careers in fields such as science, engineering, economics, and some areas of business, including those students who will be required to take a university-level calculus, linear algebra, or physics course.

**Note:** The new Advanced Functions course (MHF4U) must be taken prior to or concurrently with Calculus and Vectors (MCV4U).

## MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

- Problem Solving**
  - develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;
- Reasoning and Proving**
  - develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;
  - demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);
- Reflecting**
- Selecting Tools and Computational Strategies**
  - select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;
- Connecting**
  - make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);
  - create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;
- Representing**
- Communicating**
  - communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

# A. RATE OF CHANGE

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of rate of change by making connections between average rate of change over an interval and instantaneous rate of change at a point, using the slopes of secants and tangents and the concept of the limit;
2. graph the derivatives of polynomial, sinusoidal, and exponential functions, and make connections between the numeric, graphical, and algebraic representations of a function and its derivative;
3. verify graphically and algebraically the rules for determining derivatives; apply these rules to determine the derivatives of polynomial, sinusoidal, exponential, rational, and radical functions, and simple combinations of functions; and solve related problems.

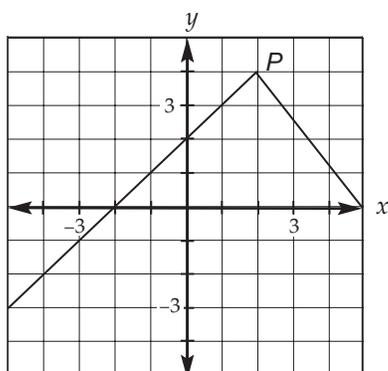
## SPECIFIC EXPECTATIONS

### 1. Investigating Instantaneous Rate of Change at a Point

By the end of this course, students will:

- 1.1 describe examples of real-world applications of rates of change, represented in a variety of ways (e.g., in words, numerically, graphically, algebraically)
- 1.2 describe connections between the average rate of change of a function that is smooth (i.e., continuous with no corners) over an interval and the slope of the corresponding secant, and between the instantaneous rate of change of a smooth function at a point and the slope of the tangent at that point

**Sample problem:** Given the graph of  $f(x)$  shown below, explain why the instantaneous rate of change of the function cannot be determined at point  $P$ .



- 1.3 make connections, with or without graphing technology, between an approximate value of the instantaneous rate of change at a given point on the graph of a smooth function and average rates of change over intervals containing the point (i.e., by using secants through the given point on a smooth curve to approach the tangent at that point, and determining the slopes of the approaching secants to approximate the slope of the tangent)

- 1.4 recognize, through investigation with or without technology, graphical and numerical examples of limits, and explain the reasoning involved (e.g., the value of a function approaching an asymptote, the value of the ratio of successive terms in the Fibonacci sequence)

**Sample problem:** Use appropriate technology to investigate the limiting value of the terms in the sequence  $\left(1 + \frac{1}{1}\right)^1, \left(1 + \frac{1}{2}\right)^2, \left(1 + \frac{1}{3}\right)^3, \left(1 + \frac{1}{4}\right)^4, \dots$ , and the limiting value of the series  $4 \times 1 - 4 \times \frac{1}{3} + 4 \times \frac{1}{5} - 4 \times \frac{1}{7} + 4 \times \frac{1}{9} - \dots$ .

- 1.5 make connections, for a function that is smooth over the interval  $a \leq x \leq a + h$ , between the average rate of change of the function over this interval and the value of the expression  $\frac{f(a+h) - f(a)}{h}$ , and between the instantaneous rate of change of the function at  $x = a$  and the value of the limit  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

**Sample problem:** What does the limit

$$\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = 8$$

indicate about the graph of the function  $f(x) = x^2$ ? The graph of a general function  $y = f(x)$ ?

- 1.6 compare, through investigation, the calculation of instantaneous rates of change at a point  $(a, f(a))$  for polynomial functions [e.g.,  $f(x) = x^2$ ,  $f(x) = x^3$ ], with and without

simplifying the expression  $\frac{f(a+h) - f(a)}{h}$

before substituting values of  $h$  that approach zero [e.g., for  $f(x) = x^2$  at  $x = 3$ , by determining

$$\frac{f(3+1) - f(3)}{1} = 7, \quad \frac{f(3+0.1) - f(3)}{0.1} = 6.1,$$

$$\frac{f(3+0.01) - f(3)}{0.01} = 6.01, \text{ and}$$

$$\frac{f(3+0.001) - f(3)}{0.001} = 6.001, \text{ and}$$

by first simplifying  $\frac{f(3+h) - f(3)}{h}$  as

$$\frac{(3+h)^2 - 3^2}{h} = 6 + h$$

and then substituting the same values of  $h$  to give the same results]

## 2. Investigating the Concept of the Derivative Function

By the end of this course, students will:

- 2.1 determine numerically and graphically the intervals over which the instantaneous rate of change is positive, negative, or zero for a function that is smooth over these intervals (e.g., by using graphing technology to examine the table of values and the slopes of tangents for a function whose equation is given; by examining a given graph), and describe the behaviour of the instantaneous rate of change at and between local maxima and minima

**Sample problem:** Given a smooth function for which the slope of the tangent is always positive, explain how you know that the function is increasing. Give an example of such a function.

- 2.2 generate, through investigation using technology, a table of values showing the instantaneous rate of change of a polynomial function,  $f(x)$ , for various values of  $x$  (e.g., construct a tangent to the function, measure its slope, and create a slider or animation to move the point of tangency), graph the ordered pairs, recognize that the graph represents a function called the derivative,  $f'(x)$  or  $\frac{dy}{dx}$ , and make connections between

the graphs of  $f(x)$  and  $f'(x)$  or  $y$  and  $\frac{dy}{dx}$

[e.g., when  $f(x)$  is linear,  $f'(x)$  is constant; when  $f(x)$  is quadratic,  $f'(x)$  is linear; when  $f(x)$  is cubic,  $f'(x)$  is quadratic]

**Sample problem:** Investigate, using patterning strategies and graphing technology, relationships between the equation of a polynomial function of degree no higher than 3 and the equation of its derivative.

- 2.3 determine the derivatives of polynomial functions by simplifying the algebraic expression

$$\frac{f(x+h) - f(x)}{h}$$

and then taking the limit of the simplified expression as  $h$  approaches zero

[i.e., determining  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ ]

- 2.4 determine, through investigation using technology, the graph of the derivative  $f'(x)$  or  $\frac{dy}{dx}$  of a given sinusoidal function [i.e.,  $f(x) = \sin x$ ,  $f(x) = \cos x$ ] (e.g., by generating a table of values showing the instantaneous rate of change of the function for various values of  $x$  and graphing the ordered pairs; by using dynamic geometry software to verify graphically that when  $f(x) = \sin x$ ,  $f'(x) = \cos x$ , and when  $f(x) = \cos x$ ,  $f'(x) = -\sin x$ ; by using a motion sensor to compare the displacement and velocity of a pendulum)

- 2.5 determine, through investigation using technology, the graph of the derivative  $f'(x)$  or  $\frac{dy}{dx}$  of a given exponential function [i.e.,  $f(x) = a^x$  ( $a > 0$ ,  $a \neq 1$ )] [e.g., by generating a table of values showing the instantaneous rate of change of the function for various values of  $x$  and graphing the ordered pairs; by using dynamic geometry software to verify that when  $f(x) = a^x$ ,  $f'(x) = kf(x)$ ], and make connections between the graphs of  $f(x)$  and  $f'(x)$  or  $y$  and  $\frac{dy}{dx}$  [e.g.,  $f(x)$  and  $f'(x)$  are both exponential; the ratio  $\frac{f'(x)}{f(x)}$  is constant, or  $f'(x) = kf(x)$ ;  $f'(x)$  is a vertical stretch from the  $x$ -axis of  $f(x)$ ]

**Sample problem:** Graph, with technology,  $f(x) = a^x$  ( $a > 0$ ,  $a \neq 1$ ) and  $f'(x)$  on the same set of axes for various values of  $a$  (e.g., 1.7, 2.0, 2.3, 3.0, 3.5). For each value of  $a$ ,

investigate the ratio  $\frac{f'(x)}{f(x)}$  for various values of  $x$ , and explain how you can use this ratio to determine the slopes of tangents to  $f(x)$ .

**2.6** determine, through investigation using technology, the exponential function  $f(x) = a^x$  ( $a > 0, a \neq 1$ ) for which  $f'(x) = f(x)$  (e.g., by using graphing technology to create a slider that varies the value of  $a$  in order to determine the exponential function whose graph is the same as the graph of its derivative), identify the number  $e$  to be the value of  $a$  for which  $f'(x) = f(x)$  [i.e., given  $f(x) = e^x, f'(x) = e^x$ ], and recognize that for the exponential function  $f(x) = e^x$  the slope of the tangent at any point on the function is equal to the value of the function at that point

**Sample problem:** Use graphing technology to determine an approximate value of  $e$  by graphing  $f(x) = a^x$  ( $a > 0, a \neq 1$ ) for various values of  $a$ , comparing the slope of the tangent at a point with the value of the function at that point, and identifying the value of  $a$  for which they are equal.

**2.7** recognize that the natural logarithmic function  $f(x) = \log_e x$ , also written as  $f(x) = \ln x$ , is the inverse of the exponential function  $f(x) = e^x$ , and make connections between  $f(x) = \ln x$  and  $f(x) = e^x$  [e.g.,  $f(x) = \ln x$  reverses what  $f(x) = e^x$  does; their graphs are reflections of each other in the line  $y = x$ ; the composition of the two functions,  $e^{\ln x}$  or  $\ln e^x$ , maps  $x$  onto itself, that is,  $e^{\ln x} = x$  and  $\ln e^x = x$ ]

**2.8** verify, using technology (e.g., calculator, graphing technology), that the derivative of the exponential function  $f(x) = a^x$  is  $f'(x) = a^x \ln a$  for various values of  $a$  [e.g., verifying numerically for  $f(x) = 2^x$  that  $f'(x) = 2^x \ln 2$  by using a calculator to show that  $\lim_{h \rightarrow 0} \frac{(2^h - 1)}{h}$  is  $\ln 2$  or by graphing  $f(x) = 2^x$ , determining the value of the slope and the value of the function for specific  $x$ -values, and comparing the ratio  $\frac{f'(x)}{f(x)}$  with  $\ln 2$ ]

**Sample problem:** Given  $f(x) = e^x$ , verify numerically with technology using

$$\lim_{h \rightarrow 0} \frac{(e^{x+h} - e^x)}{h} \text{ that } f'(x) = f(x) \ln e.$$

### 3. Investigating the Properties of Derivatives

By the end of this course, students will:

**3.1** verify the power rule for functions of the form  $f(x) = x^n$ , where  $n$  is a natural number [e.g., by determining the equations of the derivatives of the functions  $f(x) = x, f(x) = x^2, f(x) = x^3$ , and  $f(x) = x^4$  algebraically using

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ and graphically using slopes of tangents}]$$

**3.2** verify the constant, constant multiple, sum, and difference rules graphically and numerically [e.g., by using the function  $g(x) = kf(x)$  and comparing the graphs of  $g'(x)$  and  $kf'(x)$ ; by using a table of values to verify that  $f'(x) + g'(x) = (f + g)'(x)$ , given  $f(x) = x$  and  $g(x) = 3x$ ], and read and interpret proofs

involving  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  of the constant, constant multiple, sum, and difference rules (student reproduction of the development of the general case is not required)

**Sample problem:** The amounts of water flowing into two barrels are represented by the functions  $f(t)$  and  $g(t)$ . Explain what  $f'(t)$ ,  $g'(t)$ ,  $f'(t) + g'(t)$ , and  $(f + g)'(t)$  represent. Explain how you can use this context to verify the sum rule,  $f'(t) + g'(t) = (f + g)'(t)$ .

**3.3** determine algebraically the derivatives of polynomial functions, and use these derivatives to determine the instantaneous rate of change at a point and to determine point(s) at which a given rate of change occurs

**Sample problem:** Determine algebraically the derivative of  $f(x) = 2x^3 + 3x^2$  and the point(s) at which the slope of the tangent is 36.

**3.4** verify that the power rule applies to functions of the form  $f(x) = x^n$ , where  $n$  is a rational number [e.g., by comparing values of the slopes of tangents to the function  $f(x) = x^{\frac{1}{2}}$  with values of the derivative function determined using the power rule], and verify

algebraically the chain rule using monomial functions [e.g., by determining the same derivative for  $f(x) = (5x^3)^{\frac{1}{3}}$  by using the chain rule and by differentiating the simplified form,  $f(x) = 5^{\frac{1}{3}}x$ ] and the product rule using polynomial functions [e.g., by determining the same derivative for  $f(x) = (3x + 2)(2x^2 - 1)$  by using the product rule and by differentiating the expanded form  $f(x) = 6x^3 + 4x^2 - 3x - 2$ ]

**Sample problem:** Verify the chain rule by using the product rule to look for patterns in the derivatives of  $f(x) = x^2 + 1$ ,  $f(x) = (x^2 + 1)^2$ ,  $f(x) = (x^2 + 1)^3$ , and  $f(x) = (x^2 + 1)^4$ .

- 3.5** solve problems, using the product and chain rules, involving the derivatives of polynomial functions, sinusoidal functions, exponential functions, rational functions [e.g., by expressing  $f(x) = \frac{x^2 + 1}{x - 1}$  as the product  $f(x) = (x^2 + 1)(x - 1)^{-1}$ ], radical functions [e.g., by expressing  $f(x) = \sqrt{x^2 + 5}$  as the power  $f(x) = (x^2 + 5)^{\frac{1}{2}}$ ], and other simple combinations of functions [e.g.,  $f(x) = x \sin x$ ,  $f(x) = \frac{\sin x}{\cos x}$ ]\*

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\*The emphasis of this expectation is on the application of the derivative rules and not on the simplification of resulting complex algebraic expressions.

# B. DERIVATIVES AND THEIR APPLICATIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. make connections, graphically and algebraically, between the key features of a function and its first and second derivatives, and use the connections in curve sketching;
2. solve problems, including optimization problems, that require the use of the concepts and procedures associated with the derivative, including problems arising from real-world applications and involving the development of mathematical models.

## SPECIFIC EXPECTATIONS

### 1. Connecting Graphs and Equations of Functions and Their Derivatives

By the end of this course, students will:

- 1.1** sketch the graph of a derivative function, given the graph of a function that is continuous over an interval, and recognize points of inflection of the given function (i.e., points at which the concavity changes)

*Sample problem:* Investigate the effect on the graph of the derivative of applying vertical and horizontal translations to the graph of a given function.

- 1.2** recognize the second derivative as the rate of change of the rate of change (i.e., the rate of change of the slope of the tangent), and sketch the graphs of the first and second derivatives, given the graph of a smooth function

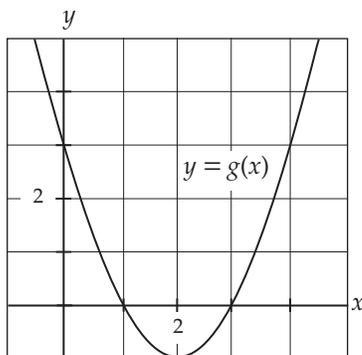
- 1.3** determine algebraically the equation of the second derivative  $f''(x)$  of a polynomial or simple rational function  $f(x)$ , and make connections, through investigation using technology, between the key features of the graph of the function (e.g., increasing/decreasing intervals, local maxima and

minima, points of inflection, intervals of concavity) and corresponding features of the graphs of its first and second derivatives (e.g., for an increasing interval of the function, the first derivative is positive; for a point of inflection of the function, the slopes of tangents change their behaviour from increasing to decreasing or from decreasing to increasing, the first derivative has a maximum or minimum, and the second derivative is zero)

*Sample problem:* Investigate, using graphing technology, connections between key properties, such as increasing/decreasing intervals, local maxima and minima, points of inflection, and intervals of concavity, of the functions  $f(x) = 4x + 1$ ,  $f(x) = x^2 + 3x - 10$ ,  $f(x) = x^3 + 2x^2 - 3x$ , and  $f(x) = x^4 + 4x^3 - 3x^2 - 18x$  and the graphs of their first and second derivatives.

- 1.4** describe key features of a polynomial function, given information about its first and/or second derivatives (e.g., the graph of a derivative, the sign of a derivative over specific intervals, the  $x$ -intercepts of a derivative), sketch two or more possible graphs of the function that are consistent with the given information, and explain why an infinite number of graphs is possible

**Sample problem:** The following is the graph of the function  $g(x)$ .



If  $g(x)$  is the derivative of  $f(x)$ , and  $f(0) = 0$ , sketch the graph of  $f(x)$ . If you are now given the function equation  $g(x) = (x - 1)(x - 3)$ , determine the equation of  $f''(x)$  and describe some features of the equation of  $f(x)$ . How would  $f(x)$  change graphically and algebraically if  $f(0) = 2$ ?

- 1.5 sketch the graph of a polynomial function, given its equation, by using a variety of strategies (e.g., using the sign of the first derivative; using the sign of the second derivative; identifying even or odd functions) to determine its key features (e.g., increasing/decreasing intervals, intercepts, local maxima and minima, points of inflection, intervals of concavity), and verify using technology

## 2. Solving Problems Using Mathematical Models and Derivatives

By the end of this course, students will:

- 2.1 make connections between the concept of motion (i.e., displacement, velocity, acceleration) and the concept of the derivative in a variety of ways (e.g., verbally, numerically, graphically, algebraically)

**Sample problem:** Generate a displacement–time graph by walking in front of a motion sensor connected to a graphing calculator. Use your knowledge of derivatives to sketch the velocity–time and acceleration–time graphs. Verify the sketches by displaying the graphs on the graphing calculator.

- 2.2 make connections between the graphical or algebraic representations of derivatives and real-world applications (e.g., population and rates of population change, prices and inflation rates, volume and rates of flow, height and growth rates)

**Sample problem:** Given a graph of prices over time, identify the periods of inflation and deflation, and the time at which the maximum rate of inflation occurred. Explain how derivatives helped solve the problem.

- 2.3 solve problems, using the derivative, that involve instantaneous rates of change, including problems arising from real-world applications (e.g., population growth, radioactive decay, temperature changes, hours of daylight, heights of tides), given the equation of a function\*

**Sample problem:** The size of a population of butterflies is given by the function

$$P(t) = \frac{6000}{1 + 49(0.6)^t} \text{ where } t \text{ is the time in days.}$$

Determine the rate of growth in the population after 5 days using the derivative, and verify graphically using technology.

- 2.4 solve optimization problems involving polynomial, simple rational, and exponential functions drawn from a variety of applications, including those arising from real-world situations

**Sample problem:** The number of bus riders from the suburbs to downtown per day is represented by  $1200(1.15)^{-x}$ , where  $x$  is the fare in dollars. What fare will maximize the total revenue?

- 2.5 solve problems arising from real-world applications by applying a mathematical model and the concepts and procedures associated with the derivative to determine mathematical results, and interpret and communicate the results

**Sample problem:** A bird is foraging for berries. If it stays too long in any one patch it will be spending valuable foraging time looking for the hidden berries, but when it leaves it will have to spend time finding another patch. A model for the net amount of food energy in

\*The emphasis of this expectation is on the application of the derivative rules and not on the simplification of resulting complex algebraic expressions.

joules the bird gets if it spends  $t$  minutes in a patch is  $E = \frac{3000t}{t + 4}$ . Suppose the bird takes

2 min on average to find each new patch, and spends negligible energy doing so. How long should the bird spend in a patch to maximize its average rate of energy gain over the time spent flying to a patch and foraging in it? Use and compare numeric, graphical, and algebraic strategies to solve this problem.

# C. GEOMETRY AND ALGEBRA OF VECTORS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of vectors in two-space and three-space by representing them algebraically and geometrically and by recognizing their applications;
2. perform operations on vectors in two-space and three-space, and use the properties of these operations to solve problems, including those arising from real-world applications;
3. distinguish between the geometric representations of a single linear equation or a system of two linear equations in two-space and three-space, and determine different geometric configurations of lines and planes in three-space;
4. represent lines and planes using scalar, vector, and parametric equations, and solve problems involving distances and intersections.

## SPECIFIC EXPECTATIONS

### 1. Representing Vectors Geometrically and Algebraically

By the end of this course, students will:

- 1.1** recognize a vector as a quantity with both magnitude and direction, and identify, gather, and interpret information about real-world applications of vectors (e.g., displacement, forces involved in structural design, simple animation of computer graphics, velocity determined using GPS)

*Sample problem:* Position is represented using vectors. Explain why knowing that someone is 69 km from Lindsay, Ontario, is not sufficient to identify their exact position.

- 1.2** represent a vector in two-space geometrically as a directed line segment, with directions expressed in different ways (e.g.,  $320^\circ$ ; N  $40^\circ$  W), and algebraically (e.g., using Cartesian coordinates; using polar coordinates), and recognize vectors with the same magnitude and direction but different positions as equal vectors

- 1.3** determine, using trigonometric relationships [e.g.,  $x = r\cos\theta$ ,  $y = r\sin\theta$ ,  $\theta = \tan^{-1}\left(\frac{y}{x}\right)$  or  $\tan^{-1}\left(\frac{y}{x}\right) + 180^\circ$ ,  $r = \sqrt{x^2 + y^2}$ ], the Cartesian representation of a vector in two-space given as a directed line segment, or

the representation as a directed line segment of a vector in two-space given in Cartesian form [e.g., representing the vector (8, 6) as a directed line segment]

*Sample problem:* Represent the vector with a magnitude of 8 and a direction of  $30^\circ$  anti-clockwise to the positive  $x$ -axis in Cartesian form.

- 1.4** recognize that points and vectors in three-space can both be represented using Cartesian coordinates, and determine the distance between two points and the magnitude of a vector using their Cartesian representations

### 2. Operating With Vectors

By the end of this course, students will:

- 2.1** perform the operations of addition, subtraction, and scalar multiplication on vectors represented as directed line segments in two-space, and on vectors represented in Cartesian form in two-space and three-space
- 2.2** determine, through investigation with and without technology, some properties (e.g., commutative, associative, and distributive properties) of the operations of addition, subtraction, and scalar multiplication of vectors

- 2.3 solve problems involving the addition, subtraction, and scalar multiplication of vectors, including problems arising from real-world applications

**Sample problem:** A plane on a heading of N 27° E has an air speed of 375 km/h. The wind is blowing from the south at 62 km/h. Determine the actual direction of travel of the plane and its ground speed.

- 2.4 perform the operation of dot product on two vectors represented as directed line segments (i.e., using  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$ ) and in Cartesian form (i.e., using  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$  or  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ ) in two-space and three-space, and describe applications of the dot product (e.g., determining the angle between two vectors; determining the projection of one vector onto another)

**Sample problem:** Describe how the dot product can be used to compare the work done in pulling a wagon over a given distance in a specific direction using a given force for different positions of the handle.

- 2.5 determine, through investigation, properties of the dot product (e.g., investigate whether it is commutative, distributive, or associative; investigate the dot product of a vector with itself and the dot product of orthogonal vectors)

**Sample problem:** Investigate geometrically and algebraically the relationship between the dot product of the vectors (1, 0, 1) and (0, 1, -1) and the dot product of scalar multiples of these vectors. Does this relationship apply to any two vectors? Find a vector that is orthogonal to both the given vectors.

- 2.6 perform the operation of cross product on two vectors represented in Cartesian form in three-space [i.e., using  $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$ ], determine the magnitude of the cross product (i.e., using  $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$ ), and describe applications of the cross product (e.g., determining a vector orthogonal to two given vectors; determining the turning effect [or torque] when a force is applied to a wrench at different angles)

**Sample problem:** Explain how you maximize the torque when you use a wrench and how the inclusion of a ratchet in the design of a wrench helps you to maximize the torque.

- 2.7 determine, through investigation, properties of the cross product (e.g., investigate whether it is commutative, distributive, or associative; investigate the cross product of collinear vectors)

**Sample problem:** Investigate algebraically the relationship between the cross product of the vectors  $\vec{a} = (1, 0, 1)$  and  $\vec{b} = (0, 1, -1)$  and the cross product of scalar multiples of  $\vec{a}$  and  $\vec{b}$ . Does this relationship apply to any two vectors?

- 2.8 solve problems involving dot product and cross product (e.g., determining projections, the area of a parallelogram, the volume of a parallelepiped), including problems arising from real-world applications (e.g., determining work, torque, ground speed, velocity, force)

**Sample problem:** Investigate the dot products  $\vec{a} \cdot (\vec{a} \times \vec{b})$  and  $\vec{b} \cdot (\vec{a} \times \vec{b})$  for any two vectors  $\vec{a}$  and  $\vec{b}$  in three-space. What property of the cross product  $\vec{a} \times \vec{b}$  does this verify?

### 3. Describing Lines and Planes Using Linear Equations

By the end of this course, students will:

- 3.1 recognize that the solution points  $(x, y)$  in two-space of a single linear equation in two variables form a line and that the solution points  $(x, y)$  in two-space of a system of two linear equations in two variables determine the point of intersection of two lines, if the lines are not coincident or parallel
- Sample problem:** Describe algebraically the situations in two-space in which the solution points  $(x, y)$  of a system of two linear equations in two variables do not determine a point.
- 3.2 determine, through investigation with technology (i.e., 3-D graphing software) and without technology, that the solution points  $(x, y, z)$  in three-space of a single linear equation in three variables form a plane and that the solution points  $(x, y, z)$  in three-space of a system of two linear equations in three variables form the line of intersection of two planes, if the planes are not coincident or parallel

**Sample problem:** Use spatial reasoning to compare the shapes of the solutions in three-space with the shapes of the solutions in two-space for each of the linear equations  $x = 0$ ,

$y = 0$ , and  $y = x$ . For each of the equations  $z = 5$ ,  $y - z = 3$ , and  $x + z = 1$ , describe the shape of the solution points  $(x, y, z)$  in three-space. Verify the shapes of the solutions in three-space using technology.

- 3.3 determine, through investigation using a variety of tools and strategies (e.g., modelling with cardboard sheets and drinking straws; sketching on isometric graph paper), different geometric configurations of combinations of up to three lines and/or planes in three-space (e.g., two skew lines, three parallel planes, two intersecting planes, an intersecting line and plane); organize the configurations based on whether they intersect and, if so, how they intersect (i.e., in a point, in a line, in a plane)

#### 4. Describing Lines and Planes Using Scalar, Vector, and Parametric Equations

By the end of this course, students will:

- 4.1 recognize a scalar equation for a line in two-space to be an equation of the form  $Ax + By + C = 0$ , represent a line in two-space using a vector equation (i.e.,  $\vec{r} = \vec{r}_0 + t\vec{m}$ ) and parametric equations, and make connections between a scalar equation, a vector equation, and parametric equations of a line in two-space
- 4.2 recognize that a line in three-space cannot be represented by a scalar equation, and represent a line in three-space using the scalar equations of two intersecting planes and using vector and parametric equations (e.g., given a direction vector and a point on the line, or given two points on the line)
- Sample problem:* Represent the line passing through  $(3, 2, -1)$  and  $(0, 2, 1)$  with the scalar equations of two intersecting planes, with a vector equation, and with parametric equations.
- 4.3 recognize a normal to a plane geometrically (i.e., as a vector perpendicular to the plane) and algebraically [e.g., one normal to the plane  $3x + 5y - 2z = 6$  is  $(3, 5, -2)$ ], and determine, through investigation, some geometric properties of the plane (e.g., the direction of any normal to a plane is constant; all scalar multiples of a normal to a plane are also normals to that plane; three non-collinear points determine a plane; the resultant, or sum, of any two vectors in a plane also lies in the plane)

*Sample problem:* How does the relationship  $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$  help you determine whether three non-parallel planes intersect in a point, if  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  represent normals to the three planes?

- 4.4 recognize a scalar equation for a plane in three-space to be an equation of the form  $Ax + By + Cz + D = 0$  whose solution points make up the plane, determine the intersection of three planes represented using scalar equations by solving a system of three linear equations in three unknowns algebraically (e.g., by using elimination or substitution), and make connections between the algebraic solution and the geometric configuration of the three planes

*Sample problem:* Determine the equation of a plane  $P_3$  that intersects the planes  $P_1$ ,  $x + y + z = 1$ , and  $P_2$ ,  $x - y + z = 0$ , in a single point. Determine the equation of a plane  $P_4$  that intersects  $P_1$  and  $P_2$  in more than one point.

- 4.5 determine, using properties of a plane, the scalar, vector, and parametric equations of a plane
- Sample problem:* Determine the scalar, vector, and parametric equations of the plane that passes through the points  $(3, 2, 5)$ ,  $(0, -2, 2)$ , and  $(1, 3, 1)$ .
- 4.6 determine the equation of a plane in its scalar, vector, or parametric form, given another of these forms
- Sample problem:* Represent the plane  $\vec{r} = (2, 1, 0) + s(1, -1, 3) + t(2, 0, -5)$ , where  $s$  and  $t$  are real numbers, with a scalar equation.
- 4.7 solve problems relating to lines and planes in three-space that are represented in a variety of ways (e.g., scalar, vector, parametric equations) and involving distances (e.g., between a point and a plane; between two skew lines) or intersections (e.g., of two lines, of a line and a plane), and interpret the result geometrically

*Sample problem:* Determine the intersection of the perpendicular line drawn from the point  $A(-5, 3, 7)$  to the plane  $\vec{v} = (0, 0, 2) + t(-1, 1, 3) + s(2, 0, -3)$ , and determine the distance from point  $A$  to the plane.

# Mathematics of Data Management, Grade 12

University Preparation

MDM4U

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This course broadens students' understanding of mathematics as it relates to managing data. Students will apply methods for organizing and analysing large amounts of information; solve problems involving probability and statistics; and carry out a culminating investigation that integrates statistical concepts and skills. Students will also refine their use of the mathematical processes necessary for success in senior mathematics. Students planning to enter university programs in business, the social sciences, and the humanities will find this course of particular interest.

**Prerequisite:** Functions, Grade 11, University Preparation, or Functions and Applications, Grade 11, University/College Preparation

## MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

### Problem Solving

- develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;

### Reasoning and Proving

- develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;

### Reflecting

- demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);

### Selecting Tools and Computational Strategies

- select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;

### Connecting

- make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);

### Representing

- create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;

### Communicating

- communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

# A. COUNTING AND PROBABILITY

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. solve problems involving the probability of an event or a combination of events for discrete sample spaces;
2. solve problems involving the application of permutations and combinations to determine the probability of an event.

## SPECIFIC EXPECTATIONS

### 1. Solving Probability Problems Involving Discrete Sample Spaces

By the end of this course, students will:

- 1.1 recognize and describe how probabilities are used to represent the likelihood of a result of an experiment (e.g., spinning spinners; drawing blocks from a bag that contains different-coloured blocks; playing a game with number cubes; playing Aboriginal stick-and-stone games) and the likelihood of a real-world event (e.g., that it will rain tomorrow, that an accident will occur, that a product will be defective)
  - 1.2 describe a sample space as a set that contains all possible outcomes of an experiment, and distinguish between a discrete sample space as one whose outcomes can be counted (e.g., all possible outcomes of drawing a card or tossing a coin) and a continuous sample space as one whose outcomes can be measured (e.g., all possible outcomes of the time it takes to complete a task or the maximum distance a ball can be thrown)
  - 1.3 determine the theoretical probability,  $P_i$  (i.e., a value from 0 to 1), of each outcome of a discrete sample space (e.g., in situations in which all outcomes are equally likely), recognize that the sum of the probabilities of the outcomes is 1 (i.e., for  $n$  outcomes,  $P_1 + P_2 + P_3 + \dots + P_n = 1$ ), recognize that the probabilities  $P_i$  form the probability distribution associated with the sample space, and solve related problems
- Sample problem:* An experiment involves rolling two number cubes and determining the sum. Calculate the theoretical probability of each outcome, and verify that the sum of the probabilities is 1.
- 1.4 determine, through investigation using class-generated data and technology-based simulation models (e.g., using a random-number generator on a spreadsheet or on a graphing calculator; using dynamic statistical software to simulate repeated trials in an experiment), the tendency of experimental probability to approach theoretical probability as the number of trials in an experiment increases (e.g., “If I simulate tossing two coins 1000 times using technology, the experimental probability that I calculate for getting two tails on the two tosses is likely to be closer to the theoretical probability of  $\frac{1}{4}$  than if I simulate tossing the coins only 10 times”)
- Sample problem:* Calculate the theoretical probability of rolling a 2 on a single roll of a number cube. Simulate rolling a number cube, and use the simulation results to calculate the experimental probabilities of rolling a 2 over 10, 20, 30, ..., 200 trials. Graph the experimental probabilities versus the number of trials, and describe any trend.
- 1.5 recognize and describe an event as a set of outcomes and as a subset of a sample space, determine the complement of an event, determine whether two or more events are mutually exclusive or non-mutually exclusive (e.g., the events of getting an even number or

getting an odd number of heads from tossing a coin 5 times are mutually exclusive), and solve related probability problems [e.g., calculate  $P(\sim A)$ ,  $P(A \text{ and } B)$ ,  $P(A \text{ or } B)$ ] using a variety of strategies (e.g., Venn diagrams, lists, formulas)

- 1.6** determine whether two events are independent or dependent and whether one event is conditional on another event, and solve related probability problems [e.g., calculate  $P(A \text{ and } B)$ ,  $P(A \text{ or } B)$ ,  $P(A \text{ given } B)$ ] using a variety of strategies (e.g., tree diagrams, lists, formulas)

## 2. Solving Problems Using Counting Principles

By the end of this course, students will:

- 2.1** recognize the use of permutations and combinations as counting techniques with advantages over other counting techniques (e.g., making a list; using a tree diagram; making a chart; drawing a Venn diagram), distinguish between situations that involve the use of permutations and those that involve the use of combinations (e.g., by considering whether or not order matters), and make connections between, and calculate, permutations and combinations
- Sample problem:* An organization with 10 members is considering two leadership models. One involves a steering committee with 4 members of equal standing. The other is an executive committee consisting of a president, vice-president, secretary, and treasurer. Determine the number of ways of selecting the executive committee from the 10 members and, using this number, the number of ways of selecting the steering committee from the 10 members. How are the calculations related? Use the calculations to explain the relationship between permutations and combinations.
- 2.2** solve simple problems using techniques for counting permutations and combinations, where all objects are distinct, and express the solutions using standard combinatorial notation [e.g.,  $n!$ ,  $P(n, r)$ ,  $\binom{n}{r}$ ]
- Sample problem:* In many Aboriginal communities, it is common practice for people to shake hands when they gather. Use combinations to determine the total number of handshakes when 7 people gather, and verify using a different strategy.
- 2.3** solve introductory counting problems involving the additive counting principle (e.g., determining the number of ways of selecting 2 boys *or* 2 girls from a group of 4 boys and 5 girls) and the multiplicative counting principle (e.g., determining the number of ways of selecting 2 boys *and* 2 girls from a group of 4 boys and 5 girls)
- 2.4** make connections, through investigation, between combinations (i.e.,  $n$  choose  $r$ ) and Pascal's triangle [e.g., between  $\binom{2}{r}$  and row 3 of Pascal's triangle, between  $\binom{n}{2}$  and diagonal 3 of Pascal's triangle]
- Sample problem:* A school is 5 blocks west and 3 blocks south of a student's home. Determine, in a variety of ways (e.g., by drawing the routes, by using Pascal's triangle, by using combinations), how many different routes the student can take from home to the school by going west or south at each corner.
- 2.5** solve probability problems using counting principles for situations involving equally likely outcomes
- Sample problem:* Two marbles are drawn randomly from a bag containing 12 green marbles and 16 red marbles. What is the probability that the two marbles are both green if the first marble is replaced? If the first marble is not replaced?

# B. PROBABILITY DISTRIBUTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of discrete probability distributions, represent them numerically, graphically, and algebraically, determine expected values, and solve related problems from a variety of applications;
2. demonstrate an understanding of continuous probability distributions, make connections to discrete probability distributions, determine standard deviations, describe key features of the normal distribution, and solve related problems from a variety of applications.

## SPECIFIC EXPECTATIONS

### 1. Understanding Probability Distributions for Discrete Random Variables

By the end of this course, students will:

- 1.1 recognize and identify a discrete random variable  $X$  (i.e., a variable that assumes a unique value for each outcome of a discrete sample space, such as the value  $x$  for the outcome of getting  $x$  heads in 10 tosses of a coin), generate a probability distribution [i.e., a function that maps each value  $x$  of a random variable  $X$  to a corresponding probability,  $P(X = x)$ ] by calculating the probabilities associated with all values of a random variable, with and without technology, and represent a probability distribution numerically using a table

- 1.2 calculate the expected value for a given probability distribution [i.e., using  $E(X) = \sum xP(X = x)$ ], interpret the expected value in applications, and make connections between the expected value and the weighted mean of the values of the discrete random variable

**Sample problem:** Of six cases, three each hold \$1, two each hold \$1000, and one holds \$100 000. Calculate the expected value and interpret its meaning. Make a conjecture about what happens to the expected value if you add \$10 000 to each case or if you multiply the amount in each case by 10. Verify your conjectures.

- 1.3 represent a probability distribution graphically using a probability histogram (i.e., a histogram on which each rectangle has a base of width 1, centred on the value of the discrete random variable, and a height equal to the probability associated with the value of the random variable), and make connections between the frequency histogram and the probability histogram (e.g., by comparing their shapes)

**Sample problem:** For the situation involving the rolling of two number cubes and determining the sum, identify the discrete random variable and generate the related probability histogram. Determine the total area of the bars in the histogram and explain your result.

- 1.4 recognize conditions (e.g., independent trials) that give rise to a random variable that follows a binomial probability distribution, calculate the probability associated with each value of the random variable, represent the distribution numerically using a table and graphically using a probability histogram, and make connections to the algebraic representation

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n - x}$$

**Sample problem:** A light-bulb manufacturer estimates that 0.5% of the bulbs manufactured are defective. Generate and graph the probability distribution for the random variable that represents the number of defective bulbs in a set of 4 bulbs.

**1.5** recognize conditions (e.g., dependent trials) that give rise to a random variable that follows a hypergeometric probability distribution, calculate the probability associated with each value of the random variable (e.g., by using a tree diagram; by using combinations), and represent the distribution numerically using a table and graphically using a probability histogram

**1.6** compare, with technology and using numeric and graphical representations, the probability distributions of discrete random variables (e.g., compare binomial distributions with the same probability of success for increasing numbers of trials; compare the shapes of a hypergeometric distribution and a binomial distribution)

*Sample problem:* Compare the probability distributions associated with drawing 0, 1, 2, or 3 face cards when a card is drawn 3 times from a standard deck with replacement (i.e., the card is replaced after each draw) and without replacement (i.e., the card is not replaced after each draw).

**1.7** solve problems involving probability distributions (e.g., uniform, binomial, hypergeometric), including problems arising from real-world applications

*Sample problem:* The probability of a business person cancelling a reservation at La Place Pascal hotel is estimated to be 8%. Generate and graph the probability distribution for the discrete random variable that represents the number of business people cancelling when there are 10 reservations. Use the probability distribution to determine the probability of at least 4 of the 10 reservations being cancelled.

## 2. Understanding Probability Distributions for Continuous Random Variables

By the end of this course, students will:

**2.1** recognize and identify a continuous random variable (i.e., a variable that assumes values from the infinite number of possible outcomes in a continuous sample space), and distinguish between situations that give rise to discrete frequency distributions (e.g., counting the number of outcomes for drawing a card or tossing three coins) and situations that give rise to continuous frequency distributions

(e.g., measuring the time taken to complete a task or the maximum distance a ball can be thrown)

**2.2** recognize standard deviation as a measure of the spread of a distribution, and determine, with and without technology, the mean and standard deviation of a sample of values of a continuous random variable

**2.3** describe challenges associated with determining a continuous frequency distribution (e.g., the inability to capture all values of the variable, resulting in a need to sample; uncertainties in measured values of the variable), and recognize the need for mathematical models to represent continuous frequency distributions

**2.4** represent, using intervals, a sample of values of a continuous random variable numerically using a frequency table and graphically using a frequency histogram and a frequency polygon, recognize that the frequency polygon approximates the frequency distribution, and determine, through investigation using technology (e.g., dynamic statistical software, graphing calculator), and compare the effectiveness of the frequency polygon as an approximation of the frequency distribution for different sizes of the intervals

**2.5** recognize that theoretical probability for a continuous random variable is determined over a range of values (e.g., the probability that the life of a lightbulb is between 90 hours and 115 hours), that the probability that a continuous random variable takes any single value is zero, and that the probabilities of ranges of values form the probability distribution associated with the random variable

**2.6** recognize that the normal distribution is commonly used to model the frequency and probability distributions of continuous random variables, describe some properties of the normal distribution (e.g., the curve has a central peak; the curve is symmetric about the mean; the mean and median are equal; approximately 68% of the data values are within one standard deviation of the mean and approximately 95% of the data values are within two standard deviations of the mean), and recognize and describe situations that can be modelled using the normal distribution (e.g., birth weights of males or of females, household incomes in a neighbourhood, baseball batting averages)

**2.7** make connections, through investigation using dynamic statistical software, between the normal distribution and the binomial and hypergeometric distributions for increasing numbers of trials of the discrete distributions (e.g., recognizing that the shape of the hypergeometric distribution of the number of males on a 4-person committee selected from a group of people more closely resembles the shape of a normal distribution as the size of the group from which the committee was drawn increases)

*Sample problem:* Explain how the total area of a probability histogram for a binomial distribution allows you to predict the area under a normal probability distribution curve.

**2.8** recognize a z-score as the positive or negative number of standard deviations from the mean to a value of the continuous random variable, and solve probability problems involving normal distributions using a variety of tools and strategies (e.g., calculating a z-score and reading a probability from a table; using technology to determine a probability), including problems arising from real-world applications

*Sample problem:* The heights of 16-month-old maple seedlings are normally distributed with a mean of 32 cm and a standard deviation of 10.2 cm. What is the probability that the height of a randomly selected seedling will be between 24.0 cm and 38.0 cm?

# C. ORGANIZATION OF DATA FOR ANALYSIS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of the role of data in statistical studies and the variability inherent in data, and distinguish different types of data;
2. describe the characteristics of a good sample, some sampling techniques, and principles of primary data collection, and collect and organize data to solve a problem.

## SPECIFIC EXPECTATIONS

### 1. Understanding Data Concepts

By the end of this course, students will:

- 1.1 recognize and describe the role of data in statistical studies (e.g., the use of statistical techniques to extract or mine knowledge of relationships from data), describe examples of applications of statistical studies (e.g., in medical research, political decision making, market research), and recognize that conclusions drawn from statistical studies of the same relationship may differ (e.g., conclusions about the effect of increasing jail sentences on crime rates)
- 1.2 recognize and explain reasons why variability is inherent in data (e.g., arising from limited accuracy in measurement or from variations in the conditions of an experiment; arising from differences in samples in a survey), and distinguish between situations that involve one variable and situations that involve more than one variable

**Sample problem:** Use the Census at School database to investigate variability in the median and mean of, or a proportion estimated from, equal-sized random samples of data on a topic such as the percentage of students who do not smoke or who walk to school, or the average height of people of a particular age. Compare the median and mean of, or a proportion estimated from, samples of increasing size with the median and mean of the population or the population proportion.

- 1.3 distinguish different types of statistical data (i.e., discrete from continuous, qualitative from quantitative, categorical from numerical, nominal from ordinal, primary from secondary, experimental from observational, microdata from aggregate data) and give examples (e.g., distinguish experimental data used to compare the effectiveness of medical treatments from observational data used to examine the relationship between obesity and type 2 diabetes or between ethnicity and type 2 diabetes)

### 2. Collecting and Organizing Data

By the end of this course, students will:

- 2.1 determine and describe principles of primary data collection (e.g., the need for randomization, replication, and control in experimental studies; the need for randomization in sample surveys) and criteria that should be considered in order to collect reliable primary data (e.g., the appropriateness of survey questions; potential sources of bias; sample size)
- 2.2 explain the distinction between the terms *population* and *sample*, describe the characteristics of a good sample, explain why sampling is necessary (e.g., time, cost, or physical constraints), and describe and compare some sampling techniques (e.g., simple random, systematic, stratified, convenience, voluntary)

**Sample problem:** What are some factors that a manufacturer should consider when determining whether to test a sample or the entire population to ensure the quality of a product?

**2.3** describe how the use of random samples with a bias (e.g., response bias, measurement bias, non-response bias, sampling bias) or the use of non-random samples can affect the results of a study

**2.4** describe characteristics of an effective survey (e.g., by giving consideration to ethics, privacy, the need for honest responses, and possible sources of bias, including cultural bias), and design questionnaires (e.g., for determining if there is a relationship between a person's age and their hours per week of Internet use, between marks and hours of study, or between income and years of education) or experiments (e.g., growth of plants under different conditions) for gathering data

**Sample problem:** Give examples of concerns that could arise from an ethical review of surveys generated by students in your school.

**2.5** collect data from primary sources, through experimentation, or from secondary sources (e.g., by using the Internet to access reliable data from a well-organized database such as E-STAT; by using print sources such as newspapers and magazines), and organize data with one or more attributes (e.g., organize data about a music collection classified by artist, date of recording, and type of music using dynamic statistical software or a spreadsheet) to answer a question or solve a problem

## D. STATISTICAL ANALYSIS

### OVERALL EXPECTATIONS

By the end of this course, students will:

1. analyse, interpret, and draw conclusions from one-variable data using numerical and graphical summaries;
2. analyse, interpret, and draw conclusions from two-variable data using numerical, graphical, and algebraic summaries;
3. demonstrate an understanding of the applications of data management used by the media and the advertising industry and in various occupations.

### SPECIFIC EXPECTATIONS

#### 1. Analysing One-Variable Data

By the end of this course, students will:

- 1.1 recognize that the analysis of one-variable data involves the frequencies associated with one attribute, and determine, using technology, the relevant numerical summaries (i.e., mean, median, mode, range, interquartile range, variance, and standard deviation)
- 1.2 determine the positions of individual data points within a one-variable data set using quartiles, percentiles, and z-scores, use the normal distribution to model suitable one-variable data sets, and recognize these processes as strategies for one-variable data analysis
- 1.3 generate, using technology, the relevant graphical summaries of one-variable data (e.g., circle graphs, bar graphs, histograms, stem-and-leaf plots, boxplots) based on the type of data provided (e.g., categorical, ordinal, quantitative)
- 1.4 interpret, for a normally distributed population, the meaning of a statistic qualified by a statement describing the margin of error and the confidence level (e.g., the meaning of a statistic that is accurate to within 3 percentage points, 19 times out of 20), and make connections, through investigation using technology (e.g., dynamic statistical software), between the sample size, the margin of error, and the

confidence level (e.g., larger sample sizes create higher confidence levels for a given margin of error)

*Sample problem:* Use census data from Statistics Canada to investigate, using dynamic statistical software, the minimum sample size such that the proportion of the sample opting for a particular consumer or voting choice is within 3 percentage points of the proportion of the population, 95% of the time (i.e., 19 times out of 20).

- 1.5 interpret statistical summaries (e.g., graphical, numerical) to describe the characteristics of a one-variable data set and to compare two related one-variable data sets (e.g., compare the lengths of different species of trout; compare annual incomes in Canada and in a third-world country; compare Aboriginal and non-Aboriginal incomes); describe how statistical summaries (e.g., graphs, measures of central tendency) can be used to misrepresent one-variable data; and make inferences, and make and justify conclusions, from statistical summaries of one-variable data orally and in writing, using convincing arguments

#### 2. Analysing Two-Variable Data

By the end of this course, students will:

- 2.1 recognize that the analysis of two-variable data involves the relationship between two attributes, recognize the correlation coefficient

as a measure of the fit of the data to a linear model, and determine, using technology, the relevant numerical summaries (e.g., summary tables such as contingency tables; correlation coefficients)

**Sample problem:** Organize data from Statistics Canada to analyse gender differences (e.g., using contingency tables; using correlation coefficients) related to a specific set of characteristics (e.g., average income, hours of unpaid housework).

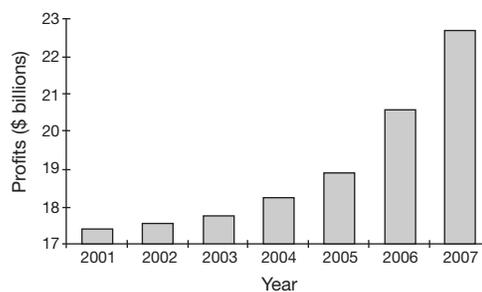
- 2.2 recognize and distinguish different types of relationships between two variables that have a mathematical correlation (e.g., the cause-and-effect relationship between the age of a tree and its diameter; the common-cause relationship between ice cream sales and forest fires over the course of a year; the accidental relationship between the consumer price index and the number of known planets in the universe)
- 2.3 generate, using technology, the relevant graphical summaries of two-variable data (e.g., scatter plots, side-by-side boxplots) based on the type of data provided (e.g., categorical, ordinal, quantitative)
- 2.4 determine, by performing a linear regression using technology, the equation of a line that models a suitable two-variable data set, determine the fit of an individual data point to the linear model (e.g., by using residuals to identify outliers), and recognize these processes as strategies for two-variable data analysis
- 2.5 interpret statistical summaries (e.g., scatter plot, equation representing a relationship) to describe the characteristics of a two-variable data set and to compare two related two-variable data sets (e.g., compare the relationship between Grade 12 English and mathematics marks with the relationship between Grade 12 science and mathematics marks); describe how statistical summaries (e.g., graphs, linear models) can be used to misrepresent two-variable data; and make inferences, and make and justify conclusions, from statistical summaries of two-variable data orally and in writing, using convincing arguments

### 3. Evaluating Validity

By the end of this course, students will:

- 3.1 interpret statistics presented in the media (e.g., the UN's finding that 2% of the world's population has more than half the world's wealth, whereas half the world's population has only 1% of the world's wealth), and explain how the media, the advertising industry, and others (e.g., marketers, pollsters) use and misuse statistics (e.g., as represented in graphs) to promote a certain point of view (e.g., by making a general statement based on a weak correlation or an assumed cause-and-effect relationship; by starting the vertical scale at a value other than zero; by making statements using general population statistics without reference to data specific to minority groups)
- 3.2 assess the validity of conclusions presented in the media by examining sources of data, including Internet sources (i.e., to determine whether they are authoritative, reliable, unbiased, and current), methods of data collection, and possible sources of bias (e.g., sampling bias, non-response bias, cultural bias in a survey question), and by questioning the analysis of the data (e.g., whether there is any indication of the sample size in the analysis) and conclusions drawn from the data (e.g., whether any assumptions are made about cause and effect)

**Sample problem:** The headline that accompanies the following graph says "Big Increase in Profits". Suggest reasons why this headline may or may not be true.



- 3.3 gather, interpret, and describe information about applications of data management in occupations (e.g., actuary, statistician, business analyst, sociologist, medical doctor, psychologist, teacher, community planner), and about university programs that explore these applications

# E. CULMINATING DATA MANAGEMENT INVESTIGATION

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. design and carry out a culminating investigation\* that requires the integration and application of the knowledge and skills related to the expectations of this course;
2. communicate the findings of a culminating investigation and provide constructive critiques of the investigations of others.

## SPECIFIC EXPECTATIONS

### 1. Designing and Carrying Out a Culminating Investigation

By the end of this course, students will:

- 1.1 pose a significant problem of interest that requires the organization and analysis of a suitable set of primary or secondary quantitative data (e.g., primary data collected from a student-designed game of chance, secondary data from a reliable source such as E-STAT), and conduct appropriate background research related to the topic being studied
- 1.2 design a plan to study the problem (e.g., identify the variables and the population; develop an ethical survey; establish the procedures for gathering, summarizing, and analysing the primary or secondary data; consider the sample size and possible sources of bias)
- 1.3 gather data related to the study of the problem (e.g., by using a survey; by using the Internet; by using a simulation) and organize the data (e.g., by setting up a database; by establishing intervals), with or without technology
- 1.4 interpret, analyse, and summarize data related to the study of the problem (e.g., generate and interpret numerical and graphical statistical summaries; recognize and apply a probability distribution model; calculate the expected value of a probability distribution), with or without technology
- 1.5 draw conclusions from the analysis of the data (e.g., determine whether the analysis solves the problem), evaluate the strength of the evidence (e.g., by considering factors such as sample size or bias, or the number of times a game is played), specify any limitations of the conclusions, and suggest follow-up problems or investigations

### 2. Presenting and Critiquing the Culminating Investigation

By the end of this course, students will:

- 2.1 compile a clear, well-organized, and detailed report of the investigation
- 2.2 present a summary of the culminating investigation to an audience of their peers within a specified length of time, with technology (e.g. presentation software) or without technology
- 2.3 answer questions about the culminating investigation and respond to critiques (e.g., by elaborating on the procedures; by justifying mathematical reasoning)
- 2.4 critique the mathematical work of others in a constructive manner

\*This culminating investigation allows students to demonstrate their knowledge and skills from this course by addressing a single problem on probability and statistics or by addressing two smaller problems, one on probability and the other on statistics.

# Mathematics for College Technology, Grade 12

College Preparation

MCT4C

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This course enables students to extend their knowledge of functions. Students will investigate and apply properties of polynomial, exponential, and trigonometric functions; continue to represent functions numerically, graphically, and algebraically; develop facility in simplifying expressions and solving equations; and solve problems that address applications of algebra, trigonometry, vectors, and geometry. Students will reason mathematically and communicate their thinking as they solve multi-step problems. This course prepares students for a variety of college technology programs.

**Prerequisite:** Functions and Applications, Grade 11, University/College Preparation, or Functions, Grade 11, University Preparation

## MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

### Problem Solving

- develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;

### Reasoning and Proving

- develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;

### Reflecting

- demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);

### Selecting Tools and Computational Strategies

- select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;

### Connecting

- make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);

### Representing

- create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;

### Communicating

- communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

# A. EXPONENTIAL FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. solve problems involving exponential equations graphically, including problems arising from real-world applications;
2. solve problems involving exponential equations algebraically using common bases and logarithms, including problems arising from real-world applications.

## SPECIFIC EXPECTATIONS

### 1. Solving Exponential Equations Graphically

By the end of this course, students will:

- 1.1 determine, through investigation with technology, and describe the impact of changing the base and changing the sign of the exponent on the graph of an exponential function
- 1.2 solve simple exponential equations numerically and graphically, with technology (e.g., use systematic trial with a scientific calculator to determine the solution to the equation  $1.05^x = 1,276$ ), and recognize that the solutions may not be exact
- 1.3 determine, through investigation using graphing technology, the point of intersection of the graphs of two exponential functions (e.g.,  $y = 4^{-x}$  and  $y = 8^{x+3}$ ), recognize the  $x$ -coordinate of this point to be the solution to the corresponding exponential equation (e.g.,  $4^{-x} = 8^{x+3}$ ), and solve exponential equations graphically (e.g., solve  $2^{x+2} = 2^x + 12$  by using the intersection of the graphs of  $y = 2^{x+2}$  and  $y = 2^x + 12$ )

**Sample problem:** Use the graph of  $y = 3^x$  to solve the equation  $3^x = 5$ .

**Sample problem:** Solve  $0.5^x = 3^{x+3}$  graphically.

- 1.4 pose problems based on real-world applications (e.g., compound interest, population growth) that can be modelled with exponential equations, and solve these and other such problems by using a given graph or a graph generated with technology from a table of values or from its equation

**Sample problem:** A tire with a slow puncture loses pressure at the rate of 4%/min. If the tire's pressure is 300 kPa to begin with, what is its pressure after 1 min? After 2 min? After 10 min? Use graphing technology to determine when the tire's pressure will be 200 kPa.

### 2. Solving Exponential Equations Algebraically

By the end of this course, students will:

- 2.1 simplify algebraic expressions containing integer and rational exponents using the laws of exponents (e.g.,  $x^3 \div x^2$ ,  $\sqrt{x^6 y^{12}}$ )
- 2.2 solve exponential equations in one variable by determining a common base (e.g.,  $2^x = 32$ ,  $4^{5x-1} = 2^{2(x+11)}$ ,  $3^{5x+8} = 27^x$ )

**Sample problem:** Simplify  $\frac{a^3 b^2 c^3}{\sqrt{a^2 b^4}}$  and then

evaluate for  $a = 4$ ,  $b = 9$ , and  $c = -3$ . Verify your answer by evaluating the expression without simplifying first. Which method for evaluating the expression do you prefer? Explain.

**Sample problem:** Solve  $3^{5x+8} = 27^x$  by determining a common base, verify by substitution, and investigate connections to the intersection of  $y = 3^{5x+8}$  and  $y = 27^x$  using graphing technology.

- 2.3 recognize the logarithm of a number to a given base as the exponent to which the base must be raised to get the number, recognize the operation of finding the logarithm to be the inverse operation (i.e., the undoing or reversing) of exponentiation, and evaluate simple logarithmic expressions

**Sample problem:** Why is it possible to determine  $\log_{10}(100)$  but not  $\log_{10}(0)$  or  $\log_{10}(-100)$ ? Explain your reasoning.

- 2.4** determine, with technology, the approximate logarithm of a number to any base, including base 10 [e.g., by recognizing that  $\log_{10}(0.372)$  can be determined using the LOG key on a calculator; by reasoning that  $\log_3 29$  is between 3 and 4 and using systematic trial to determine that  $\log_3 29$  is approximately 3.07]
- 2.5** make connections between related logarithmic and exponential equations (e.g.,  $\log_5 125 = 3$  can also be expressed as  $5^3 = 125$ ), and solve simple exponential equations by rewriting them in logarithmic form (e.g., solving  $3^x = 10$  by rewriting the equation as  $\log_3 10 = x$ )

- 2.6** pose problems based on real-world applications that can be modelled with given exponential equations, and solve these and other such problems algebraically by rewriting them in logarithmic form

**Sample problem:** When a potato whose temperature is  $20^\circ\text{C}$  is placed in an oven maintained at  $200^\circ\text{C}$ , the relationship between the core temperature of the potato  $T$ , in degrees Celsius, and the cooking time  $t$ , in minutes, is modelled by the equation  $200 - T = 180(0.96)^t$ . Use logarithms to determine the time when the potato's core temperature reaches  $160^\circ\text{C}$ .

## B. POLYNOMIAL FUNCTIONS

### OVERALL EXPECTATIONS

By the end of this course, students will:

1. recognize and evaluate polynomial functions, describe key features of their graphs, and solve problems using graphs of polynomial functions;
2. make connections between the numeric, graphical, and algebraic representations of polynomial functions;
3. solve polynomial equations by factoring, make connections between functions and formulas, and solve problems involving polynomial expressions arising from a variety of applications.

### SPECIFIC EXPECTATIONS

#### 1. Investigating Graphs of Polynomial Functions

By the end of this course, students will:

- 1.1** recognize a polynomial expression (i.e., a series of terms where each term is the product of a constant and a power of  $x$  with a non-negative integral exponent, such as  $x^3 - 5x^2 + 2x - 1$ ); recognize the equation of a polynomial function and give reasons why it is a function, and identify linear and quadratic functions as examples of polynomial functions

- 1.2** compare, through investigation using graphing technology, the graphical and algebraic representations of polynomial (i.e., linear, quadratic, cubic, quartic) functions (e.g., investigate the effect of the degree of a polynomial function on the shape of its graph and the maximum number of  $x$ -intercepts; investigate the effect of varying the sign of the leading coefficient on the end behaviour of the function for very large positive or negative  $x$ -values)

*Sample problem:* Investigate the maximum number of  $x$ -intercepts for linear, quadratic, cubic, and quartic functions using graphing technology.

- 1.3** describe key features of the graphs of polynomial functions (e.g., the domain and range, the shape of the graphs, the end behaviour of the functions for very large positive or negative  $x$ -values)

*Sample problem:* Describe and compare the key features of the graphs of the functions  $f(x) = x$ ,  $f(x) = x^2$ ,  $f(x) = x^3$ , and  $f(x) = x^4$ .

- 1.4** distinguish polynomial functions from sinusoidal and exponential functions [e.g.,  $f(x) = \sin x$ ,  $f(x) = 2^x$ ], and compare and contrast the graphs of various polynomial functions with the graphs of other types of functions
- 1.5** substitute into and evaluate polynomial functions expressed in function notation, including functions arising from real-world applications

*Sample problem:* A box with no top is being made out of a 20-cm by 30-cm piece of cardboard by cutting equal squares of side length  $x$  from the corners and folding up the sides. The volume of the box is  $V = x(20 - 2x)(30 - 2x)$ . Determine the volume if the side length of each square is 6 cm. Use the graph of the polynomial function  $V(x)$  to determine the size of square that should be cut from the corners if the required volume of the box is  $1000 \text{ cm}^3$ .

- 1.6** pose problems based on real-world applications that can be modelled with polynomial functions, and solve these and other such problems by using a given graph or a graph generated with technology from a table of values or from its equation
- 1.7** recognize, using graphs, the limitations of modelling a real-world relationship using a polynomial function, and identify and explain any restrictions on the domain and range (e.g., restrictions on the height and time for a

polynomial function that models the relationship between height above the ground and time for a falling object)

**Sample problem:** The forces acting on a horizontal support beam in a house cause it to sag by  $d$  centimetres,  $x$  metres from one end of the beam. The relationship between  $d$  and  $x$  can be represented by the polynomial function

$$d(x) = \frac{1}{1850}x(1000 - 20x^2 + x^3).$$

Graph the function, using technology, and determine the domain over which the function models the relationship between  $d$  and  $x$ . Determine the length of the beam using the graph, and explain your reasoning.

## 2. Connecting Graphs and Equations of Polynomial Functions

By the end of this course, students will:

- 2.1** factor polynomial expressions in one variable, of degree no higher than four, by selecting and applying strategies (i.e., common factoring, difference of squares, trinomial factoring)

**Sample problem:** Factor:  $x^4 - 16$ ;  $x^3 - 2x^2 - 8x$ .

- 2.2** make connections, through investigation using graphing technology (e.g., dynamic geometry software), between a polynomial function given in factored form [e.g.,  $f(x) = x(x - 1)(x + 1)$ ] and the  $x$ -intercepts of its graph, and sketch the graph of a polynomial function given in factored form using its key features (e.g., by determining intercepts and end behaviour; by locating positive and negative regions using test values between and on either side of the  $x$ -intercepts)

**Sample problem:** Sketch the graphs of  $f(x) = -(x - 1)(x + 2)(x - 4)$  and  $g(x) = -(x - 1)(x + 2)(x + 2)$  and compare their shapes and the number of  $x$ -intercepts.

- 2.3** determine, through investigation using technology (e.g., graphing calculator, computer algebra systems), and describe the connection between the real roots of a polynomial equation and the  $x$ -intercepts of the graph of the corresponding polynomial function [e.g., the real roots of the equation  $x^4 - 13x^2 + 36 = 0$  are the  $x$ -intercepts of the graph of  $f(x) = x^4 - 13x^2 + 36$ ]

**Sample problem:** Describe the relationship between the  $x$ -intercepts of the graphs of linear and quadratic functions and the real

roots of the corresponding equations. Investigate, using technology, whether this relationship exists for polynomial functions of higher degree.

## 3. Solving Problems Involving Polynomial Equations

By the end of this course, students will:

- 3.1** solve polynomial equations in one variable, of degree no higher than four (e.g.,  $x^2 - 4x = 0$ ,  $x^4 - 16 = 0$ ,  $3x^2 + 5x + 2 = 0$ ), by selecting and applying strategies (i.e., common factoring; difference of squares; trinomial factoring), and verify solutions using technology (e.g., using computer algebra systems to determine the roots of the equation; using graphing technology to determine the  $x$ -intercepts of the corresponding polynomial function)

**Sample problem:** Solve  $x^3 - 2x^2 - 8x = 0$ .

- 3.2** solve problems algebraically that involve polynomial functions and equations of degree no higher than four, including those arising from real-world applications

- 3.3** identify and explain the roles of constants and variables in a given formula (e.g., a constant can refer to a known initial value or a known fixed rate; a variable changes with varying conditions)

**Sample problem:** The formula  $P = P_0 + kh$  is used to determine the pressure,  $P$  kilopascals, at a depth of  $h$  metres under water, where  $k$  kilopascals per metre is the rate of change of the pressure as the depth increases, and  $P_0$  kilopascals is the pressure at the surface. Identify and describe the roles of  $P$ ,  $P_0$ ,  $k$ , and  $h$  in this relationship, and explain your reasoning.

- 3.4** expand and simplify polynomial expressions involving more than one variable [e.g., simplify  $-2xy(3x^2y^3 - 5x^3y^2)$ ], including expressions arising from real-world applications

**Sample problem:** Expand and simplify the expression  $\pi(R + r)(R - r)$  to explain why it represents the area of a ring. Draw a diagram of the ring and identify  $R$  and  $r$ .

- 3.5** solve equations of the form  $x^n = a$  using rational exponents (e.g., solve  $x^3 = 7$  by raising both sides to the exponent  $\frac{1}{3}$ )

- 3.6** determine the value of a variable of degree no higher than three, using a formula drawn from an application, by first substituting known values and then solving for the variable, and by first isolating the variable and then substituting known values

**Sample problem:** The formula  $s = ut + \frac{1}{2}at^2$  relates the distance,  $s$ , travelled by an object to its initial velocity,  $u$ , acceleration,  $a$ , and the elapsed time,  $t$ . Determine the acceleration of a dragster that travels 500 m from rest in 15 s, by first isolating  $a$ , and then by first substituting known values. Compare and evaluate the two methods.

- 3.7** make connections between formulas and linear, quadratic, and exponential functions [e.g., recognize that the compound interest formula,  $A = P(1 + i)^n$ , is an example of an exponential function  $A(n)$  when  $P$  and  $i$  are constant, and

of a linear function  $A(P)$  when  $i$  and  $n$  are constant], using a variety of tools and strategies (e.g., comparing the graphs generated with technology when different variables in a formula are set as constants)

**Sample problem:** Which variable(s) in the formula  $V = \pi r^2 h$  would you need to set as a constant to generate a linear equation? A quadratic equation?

- 3.8** solve multi-step problems requiring formulas arising from real-world applications (e.g., determining the cost of two coats of paint for a large cylindrical tank)
- 3.9** gather, interpret, and describe information about applications of mathematical modelling in occupations, and about college programs that explore these applications

# C. TRIGONOMETRIC FUNCTIONS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. determine the values of the trigonometric ratios for angles less than  $360^\circ$ , and solve problems using the primary trigonometric ratios, the sine law, and the cosine law;
2. make connections between the numeric, graphical, and algebraic representations of sinusoidal functions;
3. demonstrate an understanding that sinusoidal functions can be used to model some periodic phenomena, and solve related problems, including those arising from real-world applications.

## SPECIFIC EXPECTATIONS

### 1. Applying Trigonometric Ratios

By the end of this course, students will:

- 1.1 determine the exact values of the sine, cosine, and tangent of the special angles  $0^\circ$ ,  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ ,  $90^\circ$ , and their multiples
- 1.2 determine the values of the sine, cosine, and tangent of angles from  $0^\circ$  to  $360^\circ$ , through investigation using a variety of tools (e.g., dynamic geometry software, graphing tools) and strategies (e.g., applying the unit circle; examining angles related to the special angles)
- 1.3 determine the measures of two angles from  $0^\circ$  to  $360^\circ$  for which the value of a given trigonometric ratio is the same (e.g., determine one angle using a calculator and infer the other angle)

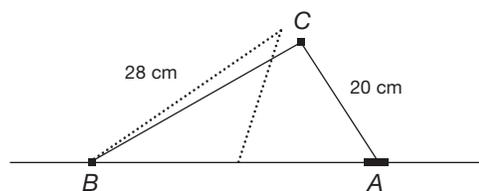
**Sample problem:** Determine the approximate measures of the angles from  $0^\circ$  to  $360^\circ$  for which the sine is 0.3423.

- 1.4 solve multi-step problems in two and three dimensions, including those that arise from real-world applications (e.g., surveying, navigation), by determining the measures of the sides and angles of right triangles using the primary trigonometric ratios

**Sample problem:** Explain how you could find the height of an inaccessible antenna on top of a tall building, using a measuring tape, a clinometer, and trigonometry. What would you measure, and how would you use the data to calculate the height of the antenna?

- 1.5 solve problems involving oblique triangles, including those that arise from real-world applications, using the sine law (including the ambiguous case) and the cosine law

**Sample problem:** The following diagram represents a mechanism in which point  $B$  is fixed, point  $C$  is a pivot, and a slider  $A$  can move horizontally as angle  $B$  changes. The minimum value of angle  $B$  is  $35^\circ$ . How far is it from the extreme left position to the extreme right position of slider  $A$ ?



### 2. Connecting Graphs and Equations of Sinusoidal Functions

By the end of this course, students will:

- 2.1 make connections between the sine ratio and the sine function and between the cosine ratio and the cosine function by graphing the relationship between angles from  $0^\circ$  to  $360^\circ$  and the corresponding sine ratios or cosine ratios, with or without technology (e.g., by generating a table of values using a calculator; by unwrapping the unit circle), defining this relationship as the function  $f(x) = \sin x$  or  $f(x) = \cos x$ , and explaining why the relationship is a function

- 2.2** sketch the graphs of  $f(x) = \sin x$  and  $f(x) = \cos x$  for angle measures expressed in degrees, and determine and describe their key properties (i.e., cycle, domain, range, intercepts, amplitude, period, maximum and minimum values, increasing/decreasing intervals)

**Sample problem:** Describe and compare the key properties of the graphs of  $f(x) = \sin x$  and  $f(x) = \cos x$ . Make some connections between the key properties of the graphs and your understanding of the sine and cosine ratios.

- 2.3** determine, through investigation using technology, the roles of the parameters  $d$  and  $c$  in functions of the form  $y = \sin(x - d) + c$  and  $y = \cos(x - d) + c$ , and describe these roles in terms of transformations on the graphs of  $f(x) = \sin x$  and  $f(x) = \cos x$  with angles expressed in degrees (i.e., vertical and horizontal translations)

**Sample problem:** Investigate the graph  $f(x) = 2 \sin(x - d) + 10$  for various values of  $d$ , using technology, and describe the effects of changing  $d$  in terms of a transformation.

- 2.4** determine, through investigation using technology, the roles of the parameters  $a$  and  $k$  in functions of the form  $y = a \sin kx$  and  $y = a \cos kx$ , and describe these roles in terms of transformations on the graphs of  $f(x) = \sin x$  and  $f(x) = \cos x$  with angles expressed in degrees (i.e., reflections in the axes; vertical and horizontal stretches and compressions to and from the  $x$ - and  $y$ -axes)

**Sample problem:** Investigate the graph  $f(x) = 2 \sin kx$  for various values of  $k$ , using technology, and describe the effects of changing  $k$  in terms of transformations.

- 2.5** determine the amplitude, period, and phase shift of sinusoidal functions whose equations are given in the form  $f(x) = a \sin(k(x - d)) + c$  or  $f(x) = a \cos(k(x - d)) + c$ , and sketch graphs of  $y = a \sin(k(x - d)) + c$  and  $y = a \cos(k(x - d)) + c$  by applying transformations to the graphs of  $f(x) = \sin x$  and  $f(x) = \cos x$

**Sample problem:** Transform the graph of  $f(x) = \cos x$  to sketch  $g(x) = 3 \cos(x + 90^\circ)$  and  $h(x) = \cos(2x) - 1$ , and state the amplitude, period, and phase shift of each function.

- 2.6** represent a sinusoidal function with an equation, given its graph or its properties

**Sample problem:** A sinusoidal function has an amplitude of 2 units, a period of  $180^\circ$ , and a maximum at  $(0, 3)$ . Represent the function with an equation in two different ways, using first the sine function and then the cosine function.

### 3. Solving Problems Involving Sinusoidal Functions

By the end of this course, students will:

- 3.1** collect data that can be modelled as a sinusoidal function (e.g., voltage in an AC circuit, pressure in sound waves, the height of a tack on a bicycle wheel that is rotating at a fixed speed), through investigation with and without technology, from primary sources, using a variety of tools (e.g., concrete materials, measurement tools such as motion sensors), or from secondary sources (e.g., websites such as Statistics Canada, E-STAT), and graph the data

**Sample problem:** Measure and record distance-time data for a swinging pendulum, using a motion sensor or other measurement tools, and graph the data. Describe how the graph would change if you moved the pendulum further away from the motion sensor. What would you do to generate a graph with a smaller amplitude?

- 3.2** identify periodic and sinusoidal functions, including those that arise from real-world applications involving periodic phenomena, given various representations (i.e., tables of values, graphs, equations), and explain any restrictions that the context places on the domain and range

**Sample problem:** The depth,  $w$  metres, of water in a lake can be modelled by the function  $w = 5 \sin(31.5n + 63) + 12$ , where  $n$  is the number of months since January 1, 1995. Identify and explain the restrictions on the domain and range of this function.

- 3.3** pose problems based on applications involving a sinusoidal function, and solve these and other such problems by using a given graph or a graph generated with technology, in degree mode, from a table of values or from its equation

**Sample problem:** The height above the ground of a rider on a Ferris wheel can be modelled by the sinusoidal function  $h(t) = 25 \cos(3(t - 60)) + 27$ , where  $h(t)$  is the height in metres and  $t$  is the time in seconds. Graph the function, using graphing technology in degree mode, and determine the maximum and minimum heights of the rider, the height after 30 s, and the time required to complete one revolution.

## D. APPLICATIONS OF GEOMETRY

### OVERALL EXPECTATIONS

By the end of this course, students will:

1. represent vectors, add and subtract vectors, and solve problems using vector models, including those arising from real-world applications;
2. solve problems involving two-dimensional shapes and three-dimensional figures and arising from real-world applications;
3. determine circle properties and solve related problems, including those arising from real-world applications.

### SPECIFIC EXPECTATIONS

#### 1. Modelling With Vectors

By the end of this course, students will:

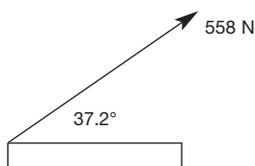
- 1.1** recognize a vector as a quantity with both magnitude and direction, and identify, gather, and interpret information about real-world applications of vectors (e.g., displacement; forces involved in structural design; simple animation of computer graphics; velocity determined using GPS)

**Sample problem:** Position is represented using vectors. Explain why knowing that someone is 69 km from Lindsay, Ontario, is not sufficient to identify their exact position.

- 1.2** represent a vector as a directed line segment, with directions expressed in different ways (e.g.,  $320^\circ$ ; N  $40^\circ$  W), and recognize vectors with the same magnitude and direction but different positions as equal vectors

- 1.3** resolve a vector represented as a directed line segment into its vertical and horizontal components

**Sample problem:** A cable exerts a force of 558 N at an angle of  $37.2^\circ$  with the horizontal. Resolve this force into its vertical and horizontal components.



- 1.4** represent a vector as a directed line segment, given its vertical and horizontal components (e.g., the displacement of a ship that travels 3 km east and 4 km north can be represented by the vector with a magnitude of 5 km and a direction of N  $36.9^\circ$  E)

- 1.5** determine, through investigation using a variety of tools (e.g., graph paper, technology) and strategies (i.e., head-to-tail method; parallelogram method; resolving vectors into their vertical and horizontal components), the sum (i.e., resultant) or difference of two vectors

- 1.6** solve problems involving the addition and subtraction of vectors, including problems arising from real-world applications (e.g., surveying, statics, orienteering)

**Sample problem:** Two people pull on ropes to haul a truck out of some mud. The first person pulls directly forward with a force of 400 N, while the other person pulls with a force of 600 N at a  $50^\circ$  angle to the first person along the horizontal plane. What is the resultant force used on the truck?

#### 2. Solving Problems Involving Geometry

By the end of this course, students will:

- 2.1** gather and interpret information about real-world applications of geometric shapes and figures in a variety of contexts in technology-related fields (e.g., product design, architecture), and explain these applications (e.g., one

reason that sewer covers are round is to prevent them from falling into the sewer during removal and replacement)

**Sample problem:** Explain why rectangular prisms are often used for packaging.

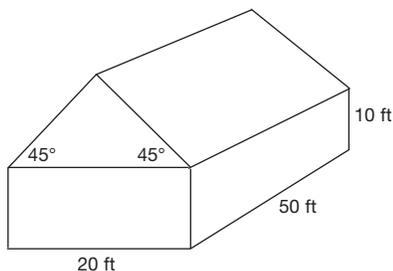
2.2 perform required conversions between the imperial system and the metric system using a variety of tools (e.g., tables, calculators, online conversion tools), as necessary within applications

2.3 solve problems involving the areas of rectangles, parallelograms, trapezoids, triangles, and circles, and of related composite shapes, in situations arising from real-world applications

**Sample problem:** Your company supplies circular cover plates for pipes. How many plates with a 1-ft radius can be made from a 4-ft by 8-ft sheet of stainless steel? What percentage of the steel will be available for recycling?

2.4 solve problems involving the volumes and surface areas of spheres, right prisms, and cylinders, and of related composite figures, in situations arising from real-world applications

**Sample problem:** For the small factory shown in the following diagram, design specifications require that the air be exchanged every 30 min. Would a ventilation system that exchanges air at a rate of  $400 \text{ ft}^3/\text{min}$  satisfy the specifications? Explain.



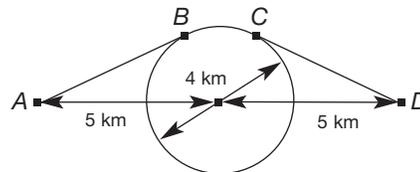
### 3. Solving Problems Involving Circle Properties

By the end of this course, students will:

3.1 recognize and describe (i.e., using diagrams and words) arcs, tangents, secants, chords, segments, sectors, central angles, and inscribed angles of circles, and some of their real-world applications (e.g., construction of a medicine wheel)

3.2 determine the length of an arc and the area of a sector or segment of a circle, and solve related problems

**Sample problem:** A circular lake has a diameter of 4 km. Points  $A$  and  $D$  are on opposite sides of the lake and lie on a straight line through the centre of the lake, with each point 5 km from the centre. In the route  $ABCD$ ,  $AB$  and  $CD$  are tangents to the lake and  $BC$  is an arc along the shore of the lake. How long is this route?

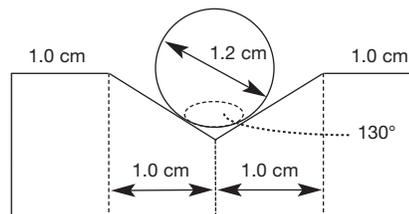


3.3 determine, through investigation using a variety of tools (e.g., dynamic geometry software), properties of the circle associated with chords, central angles, inscribed angles, and tangents (e.g., equal chords or equal arcs subtend equal central angles and equal inscribed angles; a radius is perpendicular to a tangent at the point of tangency defined by the radius, and to a chord that the radius bisects)

**Sample problem:** Investigate, using dynamic geometry software, the relationship between the lengths of two tangents drawn to a circle from a point outside the circle.

3.4 solve problems involving properties of circles, including problems arising from real-world applications

**Sample problem:** A cylindrical metal rod with a diameter of 1.2 cm is supported by a wooden block, as shown in the following diagram. Determine the distance from the top of the block to the top of the rod.



# Foundations for College Mathematics, Grade 12

College Preparation

MAP4C

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This course enables students to broaden their understanding of real-world applications of mathematics. Students will analyse data using statistical methods; solve problems involving applications of geometry and trigonometry; solve financial problems connected with annuities, budgets, and renting or owning accommodation; simplify expressions; and solve equations. Students will reason mathematically and communicate their thinking as they solve multi-step problems. This course prepares students for college programs in areas such as business, health sciences, and human services, and for certain skilled trades.

**Prerequisite:** Foundations for College Mathematics, Grade 11, College Preparation, or Functions and Applications, Grade 11, University/College Preparation

## MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

- Problem Solving**
  - develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;
- Reasoning and Proving**
  - develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;
  - demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);
- Reflecting**
- Selecting Tools and Computational Strategies**
  - select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;
- Connecting**
  - make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);
  - create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;
- Representing**
- Communicating**
  - communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

# A. MATHEMATICAL MODELS

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. evaluate powers with rational exponents, simplify algebraic expressions involving exponents, and solve problems involving exponential equations graphically and using common bases;
2. describe trends based on the interpretation of graphs, compare graphs using initial conditions and rates of change, and solve problems by modelling relationships graphically and algebraically;
3. make connections between formulas and linear, quadratic, and exponential relations, solve problems using formulas arising from real-world applications, and describe applications of mathematical modelling in various occupations.

## SPECIFIC EXPECTATIONS

### 1. Solving Exponential Equations

By the end of this course, students will:

- 1.1 determine, through investigation (e.g., by expanding terms and patterning), the exponent laws for multiplying and dividing algebraic expressions involving exponents [e.g.,  $(x^3)(x^2)$ ,  $x^3 \div x^5$ ] and the exponent law for simplifying algebraic expressions involving a power of a power [e.g.  $(x^6y^3)^2$ ]
- 1.2 simplify algebraic expressions containing integer exponents using the laws of exponents  
**Sample problem:** Simplify  $\frac{a^2b^5c^5}{ab^{-3}c^4}$  and evaluate for  $a = 8$ ,  $b = 2$ , and  $c = -30$ .
- 1.3 determine, through investigation using a variety of tools (e.g., calculator, paper and pencil, graphing technology) and strategies (e.g., patterning; finding values from a graph; interpreting the exponent laws), the value of a power with a rational exponent (i.e.,  $x^{\frac{m}{n}}$ , where  $x > 0$  and  $m$  and  $n$  are integers)  
**Sample problem:** The exponent laws suggest that  $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^1$ . What value would you assign to  $4^{\frac{1}{2}}$ ? What value would you assign to  $27^{\frac{1}{3}}$ ? Explain your reasoning. Extend your reasoning to make a generalization about the meaning of  $x^{\frac{1}{n}}$ , where  $x > 0$  and  $n$  is a natural number.
- 1.4 evaluate, with or without technology, numerical expressions involving rational exponents and rational bases [e.g.,  $2^{-3}$ ,  $(-6)^3$ ,  $4^{\frac{1}{2}}$ ,  $1.01^{120}$ ]\*
- 1.5 solve simple exponential equations numerically and graphically, with technology (e.g., use systematic trial with a scientific calculator to determine the solution to the equation  $1.05^x = 1.276$ ), and recognize that the solutions may not be exact  
**Sample problem:** Use the graph of  $y = 3^x$  to solve the equation  $3^x = 5$ .
- 1.6 solve problems involving exponential equations arising from real-world applications by using a graph or table of values generated with technology from a given equation [e.g.,  $h = 2(0.6)^n$ , where  $h$  represents the height of a bouncing ball and  $n$  represents the number of bounces]  
**Sample problem:** Dye is injected to test pancreas function. The mass,  $R$  grams, of dye remaining in a healthy pancreas after  $t$  minutes is given by the equation  $R = I(0.96)^t$ , where  $I$  grams is the mass of dye initially injected. If 0.50 g of dye is initially injected into a healthy pancreas, determine how much time elapses until 0.35 g remains by using a graph and/or table of values generated with technology.

\*The knowledge and skills described in this expectation are to be introduced as needed, and applied and consolidated, where appropriate, throughout the course.

- 1.7 solve exponential equations in one variable by determining a common base (e.g.,  $2^x = 32$ ,  $4^{5x-1} = 2^{2(x+11)}$ ,  $3^{5x+8} = 27^x$ )

**Sample problem:** Solve  $3^{5x+8} = 27^x$  by determining a common base, verify by substitution, and make connections to the intersection of  $y = 3^{5x+8}$  and  $y = 27^x$  using graphing technology.

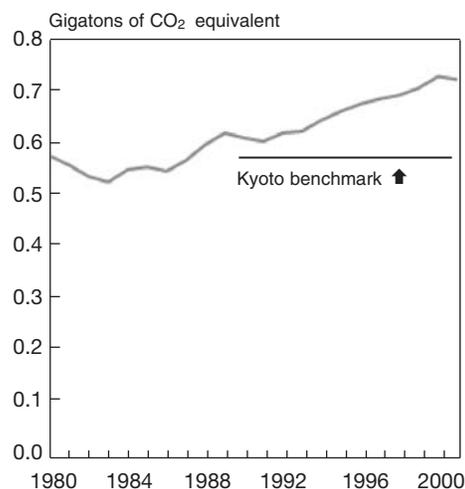
## 2. Modelling Graphically

By the end of this course, students will:

- 2.1 interpret graphs to describe a relationship (e.g., distance travelled depends on driving time, pollution increases with traffic volume, maximum profit occurs at a certain sales volume), using language and units appropriate to the context
- 2.2 describe trends based on given graphs, and use the trends to make predictions or justify decisions (e.g., given a graph of the men's 100-m world record versus the year, predict the world record in the year 2050 and state your assumptions; given a graph showing the rising trend in graduation rates among Aboriginal youth, make predictions about future rates)

**Sample problem:** Given the following graph, describe the trend in Canadian greenhouse gas emissions over the time period shown. Describe some factors that may have influenced these emissions over time. Predict the emissions today, explain your prediction using the graph and possible factors, and verify using current data.

Canadian Greenhouse Gas Emissions



Source: Environment Canada, Greenhouse Gas Inventory 1990-2001, 2003

- 2.3 recognize that graphs and tables of values communicate information about rate of change, and use a given graph or table of values for a relation to identify the units used to measure rate of change (e.g., for a distance–time graph, the units of rate of change are kilometres per hour; for a table showing earnings over time, the units of rate of change are dollars per hour)
- 2.4 identify when the rate of change is zero, constant, or changing, given a table of values or a graph of a relation, and compare two graphs by describing rate of change (e.g., compare distance–time graphs for a car that is moving at constant speed and a car that is accelerating)
- 2.5 compare, through investigation with technology, the graphs of pairs of relations (i.e., linear, quadratic, exponential) by describing the initial conditions and the behaviour of the rates of change (e.g., compare the graphs of amount versus time for equal initial deposits in simple interest and compound interest accounts)
- Sample problem:** In two colonies of bacteria, the population doubles every hour. The initial population of one colony is twice the initial population of the other. How do the graphs of population versus time compare for the two colonies? How would the graphs change if the population tripled every hour, instead of doubling?
- 2.6 recognize that a linear model corresponds to a constant increase or decrease over equal intervals and that an exponential model corresponds to a constant *percentage* increase or decrease over equal intervals, select a model (i.e., linear, quadratic, exponential) to represent the relationship between numerical data graphically and algebraically, using a variety of tools (e.g., graphing technology) and strategies (e.g., finite differences, regression), and solve related problems

**Sample problem:** Given the data table at the top of page 139, determine an algebraic model to represent the relationship between population and time, using technology. Use the algebraic model to predict the population in 2015, and describe any assumptions made.

Years after 1955	Population of Geese
0	5 000
10	12 000
20	26 000
30	62 000
40	142 000
50	260 000

### 3. Modelling Algebraically

By the end of this course, students will:

- 3.1 solve equations of the form  $x^n = a$  using rational exponents (e.g., solve  $x^3 = 7$  by raising both sides to the exponent  $\frac{1}{3}$ )
- 3.2 determine the value of a variable of degree no higher than three, using a formula drawn from an application, by first substituting known values and then solving for the variable, and by first isolating the variable and then substituting known values

*Sample problem:* Use the formula  $V = \frac{4}{3}\pi r^3$  to determine the radius of a sphere with a volume of  $1000 \text{ cm}^3$ .

- 3.3 make connections between formulas and linear, quadratic, and exponential functions [e.g., recognize that the compound interest formula,  $A = P(1 + i)^n$ , is an example of an exponential function  $A(n)$  when  $P$  and  $i$  are constant, and of a linear function  $A(P)$  when  $i$  and  $n$  are constant], using a variety of tools and strategies (e.g., comparing the graphs generated with technology when different variables in a formula are set as constants)

*Sample problem:* Which variable(s) in the formula  $V = \pi r^2 h$  would you need to set as a constant to generate a linear equation? A quadratic equation? Explain why you can expect the relationship between the volume and the height to be linear when the radius is constant.

- 3.4 solve multi-step problems requiring formulas arising from real-world applications (e.g., determining the cost of two coats of paint for a large cylindrical tank)
- 3.5 gather, interpret, and describe information about applications of mathematical modelling in occupations, and about college programs that explore these applications

## B. PERSONAL FINANCE

### OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of annuities, including mortgages, and solve related problems using technology;
2. gather, interpret, and compare information about owning or renting accommodation, and solve problems involving the associated costs;
3. design, justify, and adjust budgets for individuals and families described in case studies, and describe applications of the mathematics of personal finance.

### SPECIFIC EXPECTATIONS

#### 1. Understanding Annuities

By the end of this course, students will:

- 1.1 gather and interpret information about annuities, describe the key features of an annuity, and identify real-world applications (e.g., RRSP, mortgage, RRIF, RESP)
- 1.2 determine, through investigation using technology (e.g., the TVM Solver on a graphing calculator; online tools), the effects of changing the conditions (i.e., the payments, the frequency of the payments, the interest rate, the compounding period) of an ordinary simple annuity (i.e., an annuity in which payments are made at the *end* of each period, and compounding and payment periods are the same) (e.g., long-term savings plans, loans)
 

*Sample problem:* Given an ordinary simple annuity with semi-annual deposits of \$1000, earning 6% interest per year compounded semi-annually, over a 20-year term, which of the following results in the greatest return: doubling the payments, doubling the interest rate, doubling the frequency of the payments and the compounding, or doubling the payment and compounding period?
- 1.3 solve problems, using technology (e.g., scientific calculator, spreadsheet, graphing calculator), that involve the amount, the present value, and the regular payment of an ordinary simple annuity

*Sample problem:* Using a spreadsheet, calculate the total interest paid over the life of a \$10 000 loan with monthly repayments over 2 years at 8% per year compounded monthly, and compare the total interest with the original principal of the loan.

- 1.4 demonstrate, through investigation using technology (e.g., a TVM Solver), the advantages of starting deposits earlier when investing in annuities used as long-term savings plans

*Sample problem:* If you want to have a million dollars at age 65, how much would you have to contribute monthly into an investment that pays 7% per annum, compounded monthly, beginning at age 20? At age 35? At age 50?

- 1.5 gather and interpret information about mortgages, describe features associated with mortgages (e.g., mortgages are annuities for which the present value is the amount borrowed to purchase a home; the interest on a mortgage is compounded semi-annually but often paid monthly), and compare different types of mortgages (e.g., open mortgage, closed mortgage, variable-rate mortgage)
- 1.6 read and interpret an amortization table for a mortgage

*Sample problem:* You purchase a \$200 000 condominium with a \$25 000 down payment, and you mortgage the balance at 6.5% per year compounded semi-annually over 25 years,

payable monthly. Use a given amortization table to compare the interest paid in the first year of the mortgage with the interest paid in the 25th year.

- 1.7 generate an amortization table for a mortgage, using a variety of tools and strategies (e.g., input data into an online mortgage calculator; determine the payments using the TVM Solver on a graphing calculator and generate the amortization table using a spreadsheet), calculate the total interest paid over the life of a mortgage, and compare the total interest with the original principal of the mortgage

- 1.8 determine, through investigation using technology (e.g., TVM Solver, online tools, financial software), the effects of varying payment periods, regular payments, and interest rates on the length of time needed to pay off a mortgage and on the total interest paid

**Sample problem:** Calculate the interest saved on a \$100 000 mortgage with monthly payments, at 6% per annum compounded semi-annually, when it is amortized over 20 years instead of 25 years.

## 2. Renting or Owning Accommodation

By the end of this course, students will:

- 2.1 gather and interpret information about the procedures and costs involved in owning and in renting accommodation (e.g., apartment, condominium, townhouse, detached home) in the local community
- 2.2 compare renting accommodation with owning accommodation by describing the advantages and disadvantages of each
- 2.3 solve problems, using technology (e.g., calculator, spreadsheet), that involve the fixed costs (e.g., mortgage, insurance, property tax) and variable costs (e.g., maintenance, utilities) of owning or renting accommodation

**Sample problem:** Calculate the total of the fixed and variable monthly costs that are associated with owning a detached house but that are usually included in the rent for rental accommodation.

## 3. Designing Budgets

By the end of this course, students will::

- 3.1 gather, interpret, and describe information about living costs, and estimate the living costs of different households (e.g., a family of four, including two young children; a single young person; a single parent with one child) in the local community
- 3.2 design and present a savings plan to facilitate the achievement of a long-term goal (e.g., attending college, purchasing a car, renting or purchasing a house)
- 3.3 design, explain, and justify a monthly budget suitable for an individual or family described in a given case study that provides the specifics of the situation (e.g., income; personal responsibilities; costs such as utilities, food, rent/mortgage, entertainment, transportation, charitable contributions; long-term savings goals), with technology (e.g., using spreadsheets, budgeting software, online tools) and without technology (e.g., using budget templates)
- 3.4 identify and describe the factors to be considered in determining the affordability of accommodation in the local community (e.g., income, long-term savings, number of dependants, non-discretionary expenses), and consider the affordability of accommodation under given circumstances
 

**Sample problem:** Determine, through investigation, if it is possible to change from renting to owning accommodation in your community in five years if you currently earn \$30 000 per year, pay \$900 per month in rent, and have savings of \$20 000.
- 3.5 make adjustments to a budget to accommodate changes in circumstances (e.g., loss of hours at work, change of job, change in personal responsibilities, move to new accommodation, achievement of a long-term goal, major purchase), with technology (e.g., spreadsheet template, budgeting software)
- 3.6 gather, interpret, and describe information about applications of the mathematics of personal finance in occupations (e.g., selling real estate, bookkeeping, managing a restaurant, financial planning, mortgage brokering), and about college programs that explore these applications

# C. GEOMETRY AND TRIGONOMETRY

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. solve problems involving measurement and geometry and arising from real-world applications;
2. explain the significance of optimal dimensions in real-world applications, and determine optimal dimensions of two-dimensional shapes and three-dimensional figures;
3. solve problems using primary trigonometric ratios of acute and obtuse angles, the sine law, and the cosine law, including problems arising from real-world applications, and describe applications of trigonometry in various occupations.

## SPECIFIC EXPECTATIONS

### 1. Solving Problems Involving Measurement and Geometry

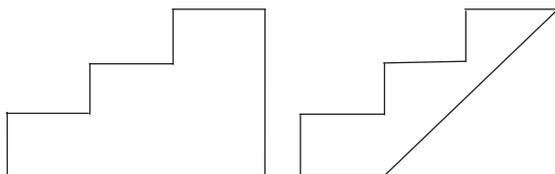
By the end of this course, students will:

- 1.1 perform required conversions between the imperial system and the metric system using a variety of tools (e.g., tables, calculators, online conversion tools), as necessary within applications
- 1.2 solve problems involving the areas of rectangles, triangles, and circles, and of related composite shapes, in situations arising from real-world applications

**Sample problem:** A car manufacturer wants to display three of its compact models in a triangular arrangement on a rotating circular platform. Calculate a reasonable area for this platform, and explain your assumptions and reasoning.

- 1.3 solve problems involving the volumes and surface areas of rectangular prisms, triangular prisms, and cylinders, and of related composite figures, in situations arising from real-world applications

**Sample problem:** Compare the volumes of concrete needed to build three steps that are 4 ft wide and that have the cross-sections shown below. Explain your assumptions and reasoning.

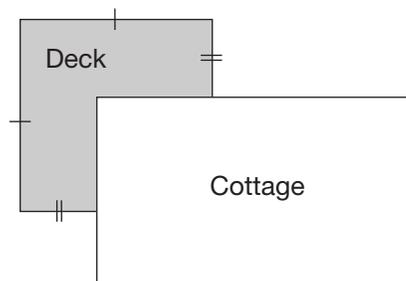


### 2. Investigating Optimal Dimensions

By the end of this course, students will:

- 2.1 recognize, through investigation using a variety of tools (e.g., calculators; dynamic geometry software; manipulatives such as tiles, geoboards, toothpicks) and strategies (e.g., modelling; making a table of values; graphing), and explain the significance of optimal perimeter, area, surface area, and volume in various applications (e.g., the minimum amount of packaging material, the relationship between surface area and heat loss)

**Sample problem:** You are building a deck attached to the second floor of a cottage, as shown below. Investigate how perimeter varies with different dimensions if you build the deck using exactly 48 1-m  $\times$  1-m decking sections, and how area varies if you use exactly 30 m of deck railing. *Note:* the entire outside edge of the deck will be railed.



**2.2** determine, through investigation using a variety of tools (e.g., calculators, dynamic geometry software, manipulatives) and strategies (e.g., modelling; making a table of values; graphing), the optimal dimensions of a two-dimensional shape in metric or imperial units for a given constraint (e.g., the dimensions that give the minimum perimeter for a given area)

**Sample problem:** You are constructing a rectangular deck against your house. You will use 32 ft of railing and will leave a 4-ft gap in the railing for access to stairs. Determine the dimensions that will maximize the area of the deck.

**2.3** determine, through investigation using a variety of tools and strategies (e.g., modelling with manipulatives; making a table of values; graphing), the optimal dimensions of a right rectangular prism, a right triangular prism, and a right cylinder in metric or imperial units for a given constraint (e.g., the dimensions that give the maximum volume for a given surface area)

**Sample problem:** Use a table of values and a graph to investigate the dimensions of a rectangular prism, a triangular prism, and a cylinder that each have a volume of  $64 \text{ cm}^3$  and the minimum surface area

### 3. Solving Problems Involving Trigonometry

By the end of this course, students will:

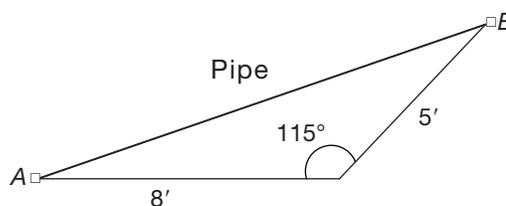
**3.1** solve problems in two dimensions using metric or imperial measurements, including problems that arise from real-world applications (e.g., surveying, navigation, building construction), by determining the measures of the sides and angles of right triangles using the primary trigonometric ratios, and of acute triangles using the sine law and the cosine law

**3.2** make connections between primary trigonometric ratios (i.e., sine, cosine, tangent) of obtuse angles and of acute angles, through investigation using a variety of tools and strategies (e.g., using dynamic geometry software to identify an obtuse angle with the same sine as a given acute angle; using a circular geoboard to compare congruent triangles; using a scientific calculator to compare trigonometric ratios for supplementary angles)

**3.3** determine the values of the sine, cosine, and tangent of obtuse angles

**3.4** solve problems involving oblique triangles, including those that arise from real-world applications, using the sine law (in non-ambiguous cases only) and the cosine law, and using metric or imperial units

**Sample problem:** A plumber must cut a piece of pipe to fit from  $A$  to  $B$ . Determine the length of the pipe.



**3.5** gather, interpret, and describe information about applications of trigonometry in occupations, and about college programs that explore these applications

**Sample problem:** Prepare a presentation to showcase an occupation that makes use of trigonometry, to describe the education and training needed for the occupation, and to highlight a particular use of trigonometry in the occupation.

## D. DATA MANAGEMENT

### OVERALL EXPECTATIONS

By the end of this course, students will:

1. collect, analyse, and summarize two-variable data using a variety of tools and strategies, and interpret and draw conclusions from the data;
2. demonstrate an understanding of the applications of data management used by the media and the advertising industry and in various occupations.

### SPECIFIC EXPECTATIONS

#### 1. Working With Two-Variable Data

By the end of this course, students will:

- 1.1** distinguish situations requiring one-variable and two-variable data analysis, describe the associated numerical summaries (e.g., tally charts, summary tables) and graphical summaries (e.g., bar graphs, scatter plots), and recognize questions that each type of analysis addresses (e.g., What is the frequency of a particular trait in a population? What is the mathematical relationship between two variables?)

*Sample problem:* Given a table showing shoe size and height for several people, pose a question that would require one-variable analysis and a question that would require two-variable analysis of the data.

- 1.2** describe characteristics of an effective survey (e.g., by giving consideration to ethics, privacy, the need for honest responses, and possible sources of bias, including cultural bias), and design questionnaires (e.g., for determining if there is a relationship between age and hours per week of Internet use, between marks and hours of study, or between income and years of education) or experiments (e.g., growth of plants under different conditions) for gathering two-variable data
- 1.3** collect two-variable data from primary sources, through experimentation involving observation or measurement, or from secondary sources (e.g., Internet databases, newspapers, magazines), and organize and store the data

using a variety of tools (e.g., spreadsheets, dynamic statistical software)

*Sample problem:* Download census data from Statistics Canada on age and average income, store the data using dynamic statistics software, and organize the data in a summary table.

- 1.4** create a graphical summary of two-variable data using a scatter plot (e.g., by identifying and justifying the dependent and independent variables; by drawing the line of best fit, when appropriate), with and without technology
- 1.5** determine an algebraic summary of the relationship between two variables that appear to be linearly related (i.e., the equation of the line of best fit of the scatter plot), using a variety of tools (e.g., graphing calculators, graphing software) and strategies (e.g., using systematic trials to determine the slope and  $y$ -intercept of the line of best fit; using the regression capabilities of a graphing calculator), and solve related problems (e.g., use the equation of the line of best fit to interpolate or extrapolate from the given data set)
- 1.6** describe possible interpretations of the line of best fit of a scatter plot (e.g., the variables are linearly related) and reasons for misinterpretations (e.g., using too small a sample; failing to consider the effect of outliers; interpolating from a weak correlation; extrapolating non-linearly related data)
- 1.7** determine whether a linear model (i.e., a line of best fit) is appropriate given a set of two-variable data, by assessing the correlation

between the two variables (i.e., by describing the type of correlation as positive, negative, or none; by describing the strength as strong or weak; by examining the context to determine whether a linear relationship is reasonable)

- 1.8** make conclusions from the analysis of two-variable data (e.g., by using a correlation to suggest a possible cause-and-effect relationship), and judge the reasonableness of the conclusions (e.g., by assessing the strength of the correlation; by considering if there are enough data)

## 2. Applying Data Management

By the end of this course, students will:

- 2.1** recognize and interpret common statistical terms (e.g., percentile, quartile) and expressions (e.g., accurate 19 times out of 20) used in the media (e.g., television, Internet, radio, newspapers)

- 2.2** describe examples of indices used by the media (e.g., consumer price index, S&P/TSX composite index, new housing price index) and solve problems by interpreting and using indices (e.g., by using the consumer price index to calculate the annual inflation rate)

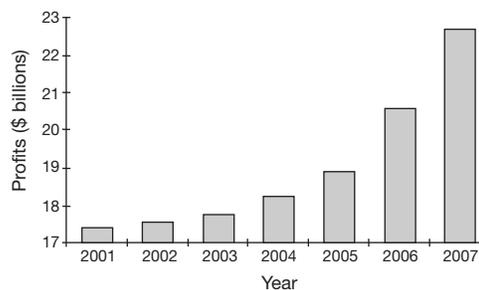
**Sample problem:** Use the new housing price index on E-STAT to track the cost of purchasing a new home over the past 10 years in the Toronto area, and compare with the cost in Calgary, Charlottetown, and Vancouver over the same period. Predict how much a new home that today costs \$200 000 in each of these cities will cost in 5 years.

- 2.3** interpret statistics presented in the media (e.g., the UN's finding that 2% of the world's population has more than half the world's wealth, whereas half the world's population has only 1% of the world's wealth), and

explain how the media, the advertising industry, and others (e.g., marketers, pollsters) use and misuse statistics (e.g., as represented in graphs) to promote a certain point of view (e.g., by making a general statement based on a weak correlation or an assumed cause-and-effect relationship; by starting the vertical scale on a graph at a value other than zero; by making statements using general population statistics without reference to data specific to minority groups)

- 2.4** assess the validity of conclusions presented in the media by examining sources of data, including Internet sources (i.e., to determine whether they are authoritative, reliable, unbiased, and current), methods of data collection, and possible sources of bias (e.g., sampling bias, non-response bias, a bias in a survey question), and by questioning the analysis of the data (e.g., whether there is any indication of the sample size in the analysis) and conclusions drawn from the data (e.g., whether any assumptions are made about cause and effect)

**Sample problem:** The headline that accompanies the following graph says "Big Increase in Profits". Suggest reasons why this headline may or may not be true.



- 2.5** gather, interpret, and describe information about applications of data management in occupations, and about college programs that explore these applications



# Mathematics for Work and Everyday Life, Grade 12

Workplace Preparation

MEL4E

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This course enables students to broaden their understanding of mathematics as it is applied in the workplace and daily life. Students will investigate questions involving the use of statistics; apply the concept of probability to solve problems involving familiar situations; investigate accommodation costs, create household budgets, and prepare a personal income tax return; use proportional reasoning; estimate and measure; and apply geometric concepts to create designs. Students will consolidate their mathematical skills as they solve problems and communicate their thinking.

**Prerequisite:** Mathematics for Work and Everyday Life, Grade 11, Workplace Preparation

## MATHEMATICAL PROCESS EXPECTATIONS

The mathematical processes are to be integrated into student learning in all areas of this course.

Throughout this course, students will:

- Problem Solving**
  - develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding;
- Reasoning and Proving**
  - develop and apply reasoning skills (e.g., use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures, assess conjectures, and justify conclusions, and plan and construct organized mathematical arguments;
  - demonstrate that they are reflecting on and monitoring their thinking to help clarify their understanding as they complete an investigation or solve a problem (e.g., by assessing the effectiveness of strategies and processes used, by proposing alternative approaches, by judging the reasonableness of results, by verifying solutions);
- Reflecting**
- Selecting Tools and Computational Strategies**
  - select and use a variety of concrete, visual, and electronic learning tools and appropriate computational strategies to investigate mathematical ideas and to solve problems;
- Connecting**
  - make connections among mathematical concepts and procedures, and relate mathematical ideas to situations or phenomena drawn from other contexts (e.g., other curriculum areas, daily life, current events, art and culture, sports);
- Representing**
  - create a variety of representations of mathematical ideas (e.g., numeric, geometric, algebraic, graphical, pictorial representations; onscreen dynamic representations), connect and compare them, and select and apply the appropriate representations to solve problems;
- Communicating**
  - communicate mathematical thinking orally, visually, and in writing, using precise mathematical vocabulary and a variety of appropriate representations, and observing mathematical conventions.

# A. REASONING WITH DATA

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. collect, organize, represent, and make inferences from data using a variety of tools and strategies, and describe related applications;
2. determine and represent probability, and identify and interpret its applications.

## SPECIFIC EXPECTATIONS

### 1. Interpreting and Displaying Data

By the end of this course, students will:

- 1.1 read and interpret graphs (e.g., bar graph, broken-line graph, histogram) obtained from various sources (e.g., newspapers, magazines, Statistics Canada website)
- 1.2 explain the distinction between the terms *population* and *sample*, describe the characteristics of a good sample, and explain why sampling is necessary (e.g., time, cost, or physical constraints)
 

**Sample problem:** What are some factors that a manufacturer should consider when determining whether to test a sample or the entire population to ensure the quality of a product?
- 1.3 collect categorical data from primary sources, through experimentation involving observation (e.g., by tracking food orders in restaurants offering healthy food options) or measurement, or from secondary sources (e.g., Internet databases, newspapers, magazines), and organize and store the data using a variety of tools (e.g., spreadsheets, dynamic statistical software)
 

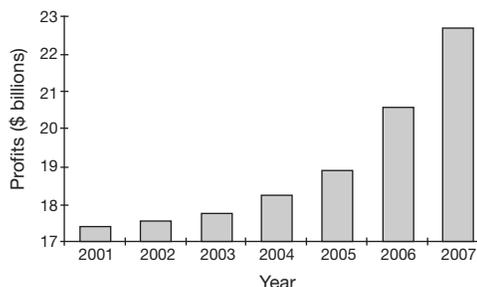
**Sample problem:** Observe cars that pass through a nearby intersection. Collect data on seatbelt usage or the number of passengers per car.
- 1.4 represent categorical data by constructing graphs (e.g., bar graph, broken-line graph, circle graph) using a variety of tools (e.g., dynamic statistical software, graphing calculator, spreadsheet)

1.5 make inferences based on the graphical representation of data (e.g., an inference about a sample from the graphical representation of a population), and justify conclusions orally or in writing using convincing arguments (e.g., by showing that it is reasonable to assume that a sample is representative of a population)

1.6 make and justify conclusions about a topic of personal interest by collecting, organizing (e.g., using spreadsheets), representing (e.g., using graphs), and making inferences from categorical data from primary sources (i.e., collected through measurement or observation) or secondary sources (e.g., electronic data from databases such as E-STAT, data from newspapers or magazines)

1.7 explain how the media, the advertising industry, and others (e.g., marketers, pollsters) use and misuse statistics (e.g., as represented in graphs) to promote a certain point of view (e.g., by making general statements based on small samples; by making statements using general population statistics without reference to data specific to minority groups)

**Sample problem:** The headline that accompanies the following graph says “Big Increase in Profits”. Suggest reasons why this headline may or may not be true.



- 1.8 gather, interpret, and describe information about applications of data management in the workplace and in everyday life

## 2. Investigating Probability

By the end of this course, students will:

- 2.1 determine the theoretical probability of an event (i.e., the ratio of the number of favourable outcomes to the total number of possible outcomes, where all outcomes are equally likely), and represent the probability in a variety of ways (e.g., as a fraction, as a percent, as a decimal in the range 0 to 1)
- 2.2 identify examples of the use of probability in the media (e.g., the probability of rain, of winning a lottery, of wait times for a service exceeding specified amounts) and various ways in which probability is represented (e.g., as a fraction, as a percent, as a decimal in the range 0 to 1)
- 2.3 perform simple probability experiments (e.g., rolling number cubes, spinning spinners, flipping coins, playing Aboriginal stick-and-stone games), record the results, and determine the experimental probability of an event
- 2.4 compare, through investigation, the theoretical probability of an event with the experimental probability, and describe how uncertainty explains why they might differ (e.g., “I know that the theoretical probability of getting tails is 0.5, but that does not mean that I will always obtain 3 tails when I toss the coin 6 times”; “If a lottery has a 1 in 9 chance of winning, am I certain to win if I buy 9 tickets?”)
- 2.5 determine, through investigation using class-generated data and technology-based simulation models (e.g., using a random-number generator on a spreadsheet or on a graphing calculator), the tendency of experimental probability to approach theoretical probability as the number of trials in an experiment increases (e.g., “If I simulate tossing a coin 1000 times using technology, the experimental probability that I calculate for getting tails in any one toss is likely to be closer to the theoretical probability than if I simulate tossing the coin only 10 times”)
- Sample problem:* Calculate the theoretical probability of rolling a 2 on a number cube. Simulate rolling a number cube, and use the simulation to calculate the experimental probability of rolling a 2 after 10, 20, 30, ..., 200 trials. Graph the experimental probability versus the number of trials, and describe any trend.
- 2.6 interpret information involving the use of probability and statistics in the media, and describe how probability and statistics can help in making informed decisions in a variety of situations (e.g., weighing the risk of injury when considering different occupations; using a weather forecast to plan outdoor activities; using sales data to stock a clothing store with appropriate styles and sizes)
- Sample problem:* A recent study on youth gambling suggests that approximately 30% of adolescents gamble on a weekly basis. Investigate and describe the assumptions that people make about the probability of winning when they gamble. Describe other factors that encourage gambling and problems experienced by people with a gambling addiction.

## B. PERSONAL FINANCE

### OVERALL EXPECTATIONS

By the end of this course, students will:

- gather, interpret, and compare information about owning or renting accommodation and about the associated costs;
- interpret, design, and adjust budgets for individuals and families described in case studies;
- demonstrate an understanding of the process of filing a personal income tax return, and describe applications of the mathematics of personal finance.

### SPECIFIC EXPECTATIONS

#### 1. Renting or Owning Accommodation

By the end of this course, students will:

- identify the financial implications (e.g., responsibility for paying the cost of accommodation and furnishings; greater responsibility for financial decision making) and the non-financial implications (e.g., greater freedom to make decisions; the demands of time management or of adapting to a new environment; the possibility of loneliness or of the need to share responsibilities) associated with living independently
- gather and compare, through investigation, information about the costs and the advantages and disadvantages of different types of rental accommodation in the local community (e.g., renting a room in someone's house; renting a hotel room; renting or leasing an apartment)
- gather and compare, through investigation, information about purchase prices of different types of owned accommodation in the local community (e.g., trailer, condominium, townhouse, detached home)
- gather, interpret, and compare information about the different types of ongoing living expenses associated with renting and owning accommodation (e.g., hydro, cable, telephone, Internet, heating, parking, laundry, groceries, cleaning supplies, transportation) and related costs

1.5 gather, interpret, and describe information about the rights and responsibilities of tenants and landlords

1.6 generate a checklist of necessary tasks associated with moving (e.g., change of address, set-up of utilities and services, truck rental), and estimate the total cost involved under various conditions (e.g., moving out of province; hiring a moving company)

#### 2. Designing Budgets

By the end of this course, students will:

- categorize personal expenses as non-discretionary (e.g., rent, groceries, utilities, loan payments) or discretionary (e.g., entertainment, vacations)
- categorize personal non-discretionary expenses as fixed (e.g., rent, cable, car insurance) or variable (e.g., groceries, clothing, vehicle maintenance)
- read and interpret prepared individual or family budgets, identify and describe the key components of a budget, and describe how budgets can reflect personal values (e.g., as they relate to shopping, saving for a long-term goal, recreational activities, family, community)
- design, with technology (e.g., using spreadsheet templates, budgeting software, online tools) and without technology (e.g., using budget templates), explain, and justify a

monthly budget suitable for an individual or family described in a given case study that provides the specifics of the situation (e.g., income; personal responsibilities; expenses such as utilities, food, rent/mortgage, entertainment, transportation, charitable contributions; long-term savings goals)

- 2.5** identify and describe factors to be considered in determining the affordability of accommodation in the local community (e.g., income, long-term savings, number of dependants, non-discretionary expenses)
- 2.6** make adjustments to a budget to accommodate changes in circumstances (e.g., loss of hours at work, change of job, change in personal responsibilities, move to new accommodation, achievement of a long-term goal, major purchase), with technology (e.g., spreadsheet template, budgeting software)

### 3. Filing Income Tax

By the end of this course, students will:

- 3.1** explain why most Canadians are expected to file a personal income tax return each year, and identify and describe the major parts of a personal income tax return (i.e., identification, total income, net income, taxable income, refund or balance owing)
- 3.2** gather, interpret, and describe the information and documents required for filing a personal income tax return (e.g., CRA guides, forms, and schedules; T4 slips; receipts for charitable donations), and explain why they are required
- 3.3** gather, interpret, and compare information about common tax credits (e.g., tuition fees, medical expenses, charitable donations) and tax deductions (e.g., moving expenses, child care expenses, union dues)
- 3.4** complete a simple personal income tax return (i.e., forms and schedules), with or without tax preparation software
- 3.5** gather, interpret, and describe some additional information that a self-employed individual should provide when filing a personal income tax return (e.g., a statement of business activities that includes business expenses such as insurance, advertising, and motor-vehicle expenses)
- 3.6** gather, interpret, and describe information about services that will complete a personal income tax return (e.g., tax preparation service, chartered accountant, voluntary service in the community) and resources that will help with completing a personal income tax return (e.g., forms and publications available on the Canada Revenue Agency website, tax preparation software for which rebates are available), and compare the services and resources on the basis of the assistance they provide and their cost
- 3.7** gather, interpret, and describe information about applications of the mathematics of personal finance in the workplace (e.g., selling real estate, bookkeeping, managing a restaurant)

# C. APPLICATIONS OF MEASUREMENT

## OVERALL EXPECTATIONS

By the end of this course, students will:

1. determine and estimate measurements using the metric and imperial systems, and convert measures within and between systems;
2. apply measurement concepts and skills to solve problems in measurement and design, to construct scale drawings and scale models, and to budget for a household improvement;
3. identify and describe situations that involve proportional relationships and the possible consequences of errors in proportional reasoning, and solve problems involving proportional reasoning, arising in applications from work and everyday life.

## SPECIFIC EXPECTATIONS

### 1. Measuring and Estimating

By the end of this course, students will:

- 1.1 measure, using a variety of tools (e.g., measuring tape, metre or yard stick, measuring cups, graduated cylinders), the lengths of common objects and the capacities of common containers, using the metric system and the imperial system
- 1.2 estimate lengths, distances, and capacities in metric units and in imperial units by applying personal referents (e.g., the width of a finger is approximately 1 cm; the length of a piece of standard loose-leaf paper is about 1 ft; the capacity of a pop bottle is 2 L)

**Sample problem:** Based on an estimate of the length of your stride, estimate how far it is to the nearest fire exit from your math classroom, and compare your estimate with the measurement you get using a pedometer.

- 1.3 estimate quantities (e.g., bricks in a pile, time to complete a job, people in a crowd), and describe the strategies used

**Sample problem:** Look at digital photos that show large quantities of items, and estimate the numbers of items in the photos.

- 1.4 convert measures within systems (e.g., centimetres and metres, kilograms and grams, litres and millilitres, feet and inches, ounces

and pounds), as required within applications that arise from familiar contexts

- 1.5 convert measures between systems (e.g., centimetres and inches, pounds and kilograms, square feet and square metres, litres and U.S. gallons, kilometres and miles, cups and millilitres, millilitres and teaspoons, degrees Celsius and degrees Fahrenheit), as required within applications that arise from familiar contexts

**Sample problem:** Compare the price of gasoline in your community with the price of gasoline in a community in the United States.

### 2. Applying Measurement and Design

By the end of this course, students will:

- 2.1 construct accurate right angles in practical contexts (e.g., by using the 3-4-5 triplet to construct a region with right-angled corners on a floor), and explain connections to the Pythagorean theorem
- 2.2 apply the concept of perimeter in familiar contexts (e.g., baseboard, fencing, door and window trim)

**Sample problem:** Which room in your home required the greatest, and which required the least, amount of baseboard? What is the difference in the two amounts?

- 2.3** estimate the areas and volumes of irregular shapes and figures, using a variety of strategies (e.g., counting grid squares; displacing water)

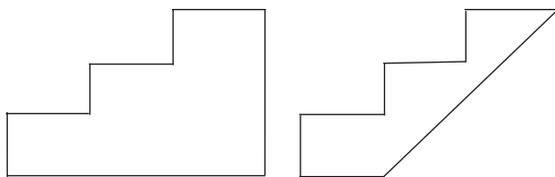
**Sample problem:** Draw an outline of your hand and estimate the area.

- 2.4** solve problems involving the areas of rectangles, triangles, and circles, and of related composite shapes, in situations arising from real-world applications

**Sample problem:** A car manufacturer wants to display three of its compact models in a triangular arrangement on a rotating circular platform. Calculate a reasonable area for this platform, and explain your assumptions and reasoning.

- 2.5** solve problems involving the volumes and surface areas of rectangular prisms, triangular prisms, and cylinders, and of related composite figures, in situations arising from real-world applications

**Sample problem:** Compare the volumes of concrete needed to build three steps that are 4 ft wide and that have the cross-sections shown below. Explain your assumptions and reasoning.



- 2.6** construct a two-dimensional scale drawing of a familiar setting (e.g., classroom, flower bed, playground) on grid paper or using design or drawing software

**Sample problem:** Your family is moving to a new house with a living room that is 16 ft by 10 ft. Cut out and label simple geometric shapes, drawn to scale, to represent every piece of furniture in your present living room. Place all of your cut-outs on a scale drawing of the new living room to find out if the furniture will fit appropriately (e.g., allowing adequate space to move around).

- 2.7** construct, with reasonable accuracy, a three-dimensional scale model of an object or environment of personal interest (e.g., appliance, room, building, garden, bridge)

**Sample problem:** Design an innovative combination of two appliances or two other consumer products (e.g., a camera and a cellphone, a refrigerator and a television), and construct a three-dimensional scale model.

- 2.8** investigate, plan, design, and prepare a budget for a household improvement (e.g., landscaping a property; renovating a room), using appropriate technologies (e.g., design or decorating websites, design or drawing software, spreadsheet)

**Sample problem:** Plan, design, and prepare a budget for the renovation of a 12-ft by 12-ft bedroom for under \$2000. The renovations could include repainting the walls, replacing the carpet with hardwood flooring, and refurbishing the room.

### 3. Solving Measurement Problems Using Proportional Reasoning

By the end of this course, students will:

- 3.1** identify and describe applications of ratio and rate, and recognize and represent equivalent ratios (e.g., show that 4:6 represents the same ratio as 2:3 by showing that a ramp with a height of 4 m and a base of 6 m and a ramp with a height of 2 m and a base of 3 m are equally steep) and equivalent rates (e.g., recognize that paying \$1.25 for 250 mL of tomato sauce is equivalent to paying \$3.75 for 750 mL of the same sauce), using a variety of tools (e.g., concrete materials, diagrams, dynamic geometry software)

- 3.2** identify situations in which it is useful to make comparisons using unit rates, and solve problems that involve comparisons of unit rates

**Sample problem:** If 500 mL of juice costs \$2.29 and 750 mL of the same juice costs \$3.59, which size is the better buy? Explain your reasoning.

- 3.3** identify and describe real-world applications of proportional reasoning (e.g., mixing concrete; calculating dosages; converting units; painting walls; calculating fuel consumption; calculating pay; enlarging patterns), distinguish between a situation involving a proportional relationship (e.g., recipes, where doubling the quantity of each ingredient doubles the number of servings; long-distance phone calls

billed at a fixed cost per minute, where talking for half as many minutes costs half as much) and a situation involving a non-proportional relationship (e.g., cellular phone packages, where doubling the minutes purchased does not double the cost of the package; food purchases, where it can be less expensive to buy the same quantity of a product in one large package than in two or more small packages; hydro bills, where doubling consumption does not double the cost) in a personal and/or workplace context, and explain their reasoning

- 3.4** identify and describe the possible consequences (e.g., overdoses of medication; seized engines; ruined clothing; cracked or crumbling concrete) of errors in proportional reasoning (e.g., not recognizing the importance of maintaining proportionality; not correctly calculating the amount of each component in a mixture)

*Sample problem:* Age, gender, body mass, body chemistry, and habits such as smoking are some factors that can influence the effectiveness of a medication. For which of these factors might doctors use proportional reasoning to adjust the dosage of medication? What are some possible consequences of making the adjustments incorrectly?

- 3.5** solve problems involving proportional reasoning in everyday life (e.g., applying fertilizers; mixing gasoline and oil for use in small engines; mixing cement; buying plants for flower beds; using pool or laundry chemicals; doubling recipes; estimating cooking time from the time needed per pound; determining the fibre content of different sizes of food servings)

*Sample problem:* Bring the label from a large can of stew to class. Use the information on the label to calculate how many calories and how much fat you would consume if you ate the whole can for dinner. Then search out information on a form of exercise you could choose for burning all those calories. For what length of time would you need to exercise?

- 3.6** solve problems involving proportional reasoning in work-related situations (e.g., calculating overtime pay; calculating pay for piecework; mixing concrete for small or large jobs)

*Sample problem:* Coiled pipe from the United States is delivered in 200-ft lengths. Your company needs pipe in 3.7-m sections. How many sections can you make from each 200-ft length?



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